The Applied Algebra Workbook

David Nacin
William Paterson University

March 22, 2018

Contents

Chapter 1

Some Basic Results in Group Theory

1.1 The Dihedral Groups

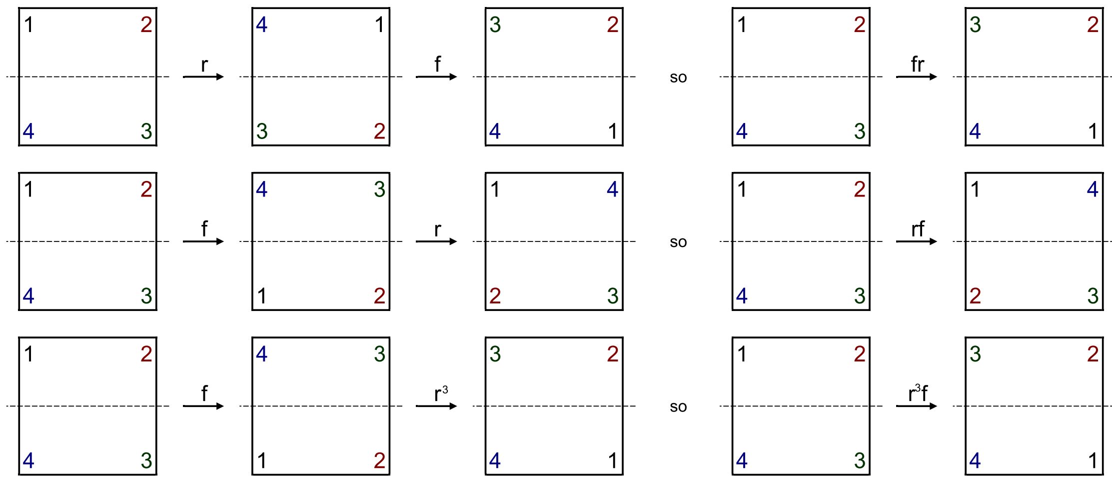
There are many different ways one can define the dihedral groups. It's helpful to first look at them as actual reflections and rotations of some object. We can define these groups by flipping and rotating this object and then making a multiplication table out of the results. This becomes impractical for large values of n, but works very well up to about D_5.

For an example, we can start by taking a college textbook, or to be precise, a two-dimensional image of one. Picture it lying face up in some plane, and picture also a line straight through the middle of the book. There are many choices for such a line, and it actually doesn't matter which we choose, but it's probably easiest to imagine one parallel to two of the edges of the book. We start by defining two group elements. The element f in D_n corresponds to the motion of flipping the book over that axis, so it is now face down with the top of the book pointing the opposite direction as before. The element r corresponds to rotating the book clockwise $360/n$ degrees through a point directly in its center. For example, r^2f is the motion we get from flipping and then rotating twice, and $frfr^2$ is the motion we get from rotating twice, flipping, rotating, and then flipping again. We perform these operations from right to left in all cases in order to match the common notation for compositions used in Pre-Calculus and Calculus, where $f \circ g$ equals $f(g(x))$ and g ends up applied first. We have an equivalence relation on the set of all the possible motions that arise from combining these operations, considering two of these motions to be equal if they affect our object in the same way. This set of equivalence classes under composition forms the dihedral group.

It helps to examine the relationship between f and r that arises because we set two of these motions to be equivalent if they leave the book in the same position. For example, in D_4, if we rotate and flip, the book is in the same position as if we had flipped first and rotated three times, and thus $fr = r^3f$. This is illustrated in figure 1.1 where we label a square book with the numbers one through four, and illustrate fr, rf and then r^3f. The dotted line depicts the axis for reflection, which does not change as the book is rotated. Every motion in D_n ends up equivalent to one of the following: $e, r, r^2, \cdots, r^{n-1}, f, rf, r^2f, \cdots, r^{n-1}f$ and thus D_n contains $2n$ elements.

It is important to keep in mind that the axis over which we flip the book does not rotate together with the object we are rotating. No matter which position the book is in, when we apply the flip, we always flip over the initial axis. It is also important to note that the elements in the group are the actual motions, and not the book itself. We use the positions to determine if two motions are equivalent, by seeing if the leave

Figure 1.1: Showing that $fr \neq rf$ and $fr = r^3 f$ in D_4



the book in the same place.

We understand that this method of first defining the dihedral groups is not particularly rigorous. For example, in D_{1000} it would be particularly difficult to see that fr^{501} is equal to $r^{499}f$. However, for small enough values of n one can get through most of the questions simply by taking a book and making these flips, and we feel it is nice to introduce these groups through an actual process that students can put their hands on.

Another way to construct D_n is by looking at it as a subgroup of the symmetric group S_n. For even n this is the smallest group containing the permutation $(1, 2, 3, \cdots, n)$ which represents r and $(1, n)(2, n-1) \cdots (n/2, (n/2)+1)$ which represents f. This will sometimes be useful when the numbers correspond to the physical corners and edges of some object. If we recall how to multiply in the symmetric group we can find the relations for our group by simply taking products. For example we can see that $rf = fr^3$ in D_4 because $(1, 2, 3, 4)(1, 4)(2, 3)$ and $(1, 4)(2, 3)(1, 2, 3, 4)^3$ both equal $(2, 4)$. There is an alternate construction for odd n where r is still $(1, 2, 3, \cdots, n)$ but f is now $(1, n)(2, n-1) \cdots ((n-1)/2, (n+1)/2)$. We will look into this construction more in the section on symmetric groups.

For these problems it is enough to consider the group presented as $D_n =< r, f | r^n = f^2 = e, fr = r^{n-1}f >$. There is more than one form for this presentation, Note that there are some prettier ways to do this, but this form has some distinct advantages. Every element can be written in the form $r^i f^j$ where $i \in \{0, 1, 2, \cdots, (n-1)\}$ and $j \in \{0, 1\}$. Then we can quickly simplify any product simply by pushing every r to the right of an f past that f, turning it into a r^{n-1}.

1.1.1 Arbitrary Dihedral Group Questions

1. Use the fact that $fr = r^{n-1}f$ to prove that $fr^k = r^{n-k}f$.
   [Answer: Moving each r past our f shows $fr^k = (r^{n-1})^k f = r^{nk-k} f = r^{n(k-1)+n-k} = (r^n)^{k-1} r^{n-k} = e^{k-1} r^{n-k} = r^{n-k}$. Note that this is equivalent to the statement $(n-1)k \equiv -k$ modulo n.]

2. Prove that for any $k \geq 0$, an element of the form $r^k f$ is its own inverse in D_n.
   [Answer: We just need to show the square of any such element is the identity. $(r^n f)^2 = (r^n f)(r^n f) =$

$r^n(r^{n-1})^n ff = r^n(r^{n(n-1)})e = r^{(n+n^2-n)} = r^{(n^2)} = (r^n)^n = e^n = e$.]

3. Prove that for odd n, D_n has n elements of order two.
[Answer: We have already shown that $r^k f$ has order two for $0 \le k < n$. This gives us n elements of order two. We must show there are no more. The identity always has order one. We have to show that no element from $r, r^2, \cdots, r^{n-1}$ is its own inverses. Take any r^k for $0 < k < n$. Then $(r^k)^2 = r^{2k}$. If this is the identity then n would have to divide $2k$. As n is odd, it would also have to divide k. This cannot happen with $0 < k < n$.]

4. Prove that for even n, D_n has $n+1$ elements of order two.
[Answer: We have already shown that $r^k f$ has order two for $0 \le k < n$. The identity has order one. We must show that only one element from $r, r^2, \cdots, r^{n-1}$ has order two. If $0 < k < n$ then $(r^k)^2 = r^{2k}$. This is only the identity if n divides $2k$. This can only happen if $k = n/2$. Alternatively, we could have looked at these as being inside the subgroup of all possible r^k. This is cyclic, so we could use the fundamental theorem of cyclic groups. This says there are $\phi(k)$ elements of order k, so long as k divides n. Thus there are $\phi(2) = 1$ element of order two amongst these.]

5. Pick a large number k of your choosing. In D_k how many elements do we have of each order?
[Answer: Large is relative. Let's try this for $k = 20$. We know any element of the form $r^k f$ for $0 \le k < n$ has order two. We know that the other ten elements for a cyclic subgroup of D_{20} which has order ten. Thus for each divisor k of ten we get $\phi(k)$ elements of that order. This gives us one element of order one, one of order two, four elements of order five and four elements of order ten. This means that an order table would look like the following:

order	1	2	3	4	5	6	7	8	9	10
#elts	1	11	0	0	4	0	0	0	0	4

.]

6. Consider the group D_p where p is an odd prime. How many elements do we have of each order?
[Answer: We know there are at least p elements of order two because of the elements $f, rf, \cdots, r^{p-1}f$. The other elements form a cyclic group of order p. We have $\phi(p) = p-1$ elements of order p. This leaves only the identity left, which has order one. We conclude that we have one element of order one, p elements of order two, and $p-1$ elements of order $p-1$.]

7. In D_n for $n > 1$, find all elements that commute with f.
[Answer: First look at which elements of the form r^k commute with f. For $1 \le k < n$, $f \times r^k = r^{n-k} f$ and $r^k \times f = r^k f$. These are only equal if r^{n-k} equals r^k. This means $n - k \equiv k$, and thus $0 \equiv 2k$ modulo n. Thus n divides $2k$. Given the range of possibilities for k, this is only possible if n is even, and only can happens when $k = n/2$. In this even case, $fr^{n/2} = r^{n-n/2}f = r^{n/2}f$ so the two actually do commute. This shows that no elements of the form $r^k, 1 \le k < n$ commute with f if n is odd, and only $r^{n/2}$ commutes with f if n is even.

Next look at elements of the form $r^k f$ that commute with f for $1 \le k < n$. Knowing the element $r^k f$ equals r^k times f, multiplying both sides of the equation by f reveals that $(r^k f) \times f = f \times (r^k f)$ if and only if $r^k f = fr^k$. Our previous computation shows this only happens when n is even and $k = n/2$.

We now can see that if n is odd, then the only elements that commute with f are e and f itself. If n is even, then the elements commuting with f are precisely $e, r^{n/2}, f$, and $r^{n/2}f$.]

8. Find the center of D_n for any value $n > 2$.
[Answer: Recall that the center of a group is the subgroup of all elements which commute with everything in the group. Anything in the center of D_n must also commute with f. We have seen that if n

is odd then the only elements commuting with f are f and e. As f does not commute with r, f is not in the center. Thus the center for odd values of n is only e.

For even values, the possibilities are $e, f, r^{n/2}$, and $r^{n/2}f$. We can rule out f and $r^{n/2}f$ by showing neither commutes with r. This leaves e and $r^{n/2}$. The latter is in the center if and only if $r^{n/2}$ commutes with all the other elements in the group, but as $e, r, r^2, \cdots, r^{n-1}$ forms a cyclic group, it automatically commutes with r^m for any m. As it commutes with r^m and f it must also commute with $r^m f$ for each m. Thus it commutes with everything and falls in the center as well. Thus the center in the even case is simply $\{e, r^{n/2}\}$.]

9. Show that for any n, D_n has a cyclic subgroup of size n.
[Answer: Simply take the group of rotations without any flips. This is $\{e, r, r^2, \cdots r^{n-1}\}$ which has size n and is generated by r.]

10. Show that for any even $n > 2$, D_n has a non-cyclic subgroup of size four.
[Answer: Since we have rotation by 180 degrees and a flip, we can take the subgroup generated by these two elements. This is the subgroup $\{e, r^{n/2}, f, r^{n/2}f\}$ which is isomorphic to D_2 and V and thus not cyclic.]

11. Show that for any even $n > 4$, D_n has a non-abelian subgroup of order n.
[Answer: Here we can consider $\{e, r^2, \cdots r^{n-2}\}$ together with the flipped versions $\{f, fr^2, \cdots, fr^{n-2}\}$ to get a copy of $D_{n/2}$ sitting inside D_n.]

1.1.2 The Group D_3

1. Find the orders of the elements of D_3.
[Answer:

element	e	r	r^2	f	rf	r^2f
order	1	3	3	2	2	2

.]

2. How many elements are there of each order in D_3?
[Answer:

order	1	2	3
#elts	1	3	2

.]

3. Find the inverses of the elements of D_3.
[Answer:

elt	e	r	r^2	f	rf	r^2f
inverse	e	r^2	r	f	rf	r^2f

.]

4. Find the center of D_3
[Answer: $\{e\}$.]

5. Compute the following products in D_3.

   (a) $r^2f \times rf$
   [Answer: r.]

   (b) $rf \times r^2$
   [Answer: r^2f.]

   (c) $rf \times r$
   [Answer: f.]

(d) $f \times rf$
[Answer: r^2.]

(e) $r^2 f \times rf \times r^2 f$
[Answer: f.]

(f) $rf \times r^2 \times f \times r$
[Answer: e.]

(g) $f \times rf \times r^2 f$
[Answer: rf.]

(h) $f \times r^2 f \times rf$
[Answer: $r^2 f$.]

6. Solve the following equations for x in D_3.

(a) $fx = r^2$
[Answer: We can multiply both sides on the left by f^{-1}, which is f itself, to get $x = f \times r^2$. This is equal to rf which is our answer.]

(b) $r^2 fx = rf$
[Answer: r.]

(c) $r^2 x = f$
[Answer: rf.]

(d) $rx = r^2 f$
[Answer: rf.]

(e) $rfx = r$
[Answer: f.]

7. Which elements do we get when we conjugate x by a for the following x and a? [Recall this means taking axa^{-1} and that this equivalent to mapping x through the inner automorphism $\phi_a(z)$.]

(a) $x = r^2, a = f$
[Answer: We get $fr^2 f^{-1} = fr^2 f = r$.]

(b) $x = rf, a = f$
[Answer: $r^2 f$.]

(c) $x = r, a = rf$
[Answer: r^2.]

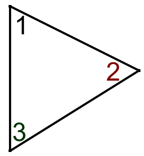
8. Make a Cayley table for the group D_3.

[Answer:

$\times$	e	r	r^2	f	rf	$r^2 f$
e	e	r	r^2	f	rf	$r^2 f$
r	r	r^2	e	rf	$r^2 f$	f
r^2	r^2	e	r	$r^2 f$	f	rf
f	f	$r^2 f$	rf	e	r^2	r
rf	rf	f	$r^2 f$	r	e	r^2
$r^2 f$	$r^2 f$	rf	f	r^2	r	e

.]

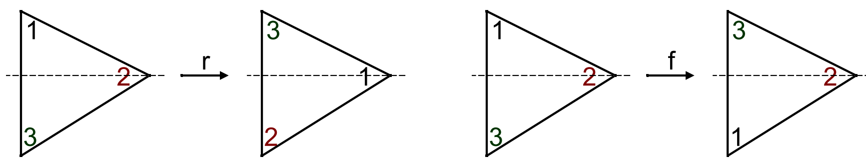
Figure 1.2: A labeled triangle.



9. Consider a triangle with its corners numbered as shown in figure 1.2.

   Let r be the map that rotates this triangle eighty degrees, and f be the map that reflects it over the axis shown and represented by the dotted line. The results of these two transformations are shown in figure 1.3.

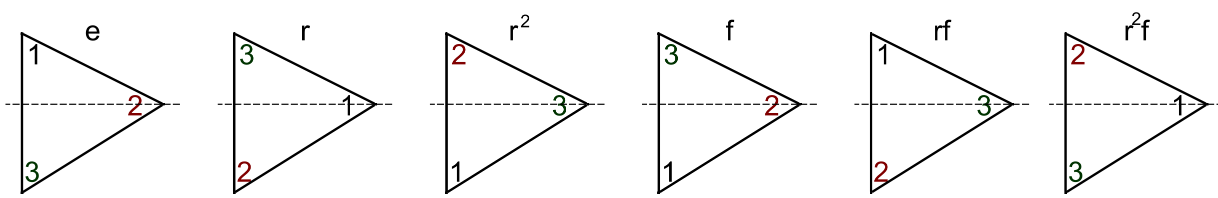
Figure 1.3: The transformations r and f.



For each element in D_3, make a diagram showing the position of the corners after the motion for that element is applied.

[Answer: See figure 1.4]

Figure 1.4: A labeled triangle after individual elements of D_3 have been applied.



10. If r represents rotation by 120 degrees, and f represents flipping over the axis shown, find the elements of $D_3 = \{e, r, r^2, f, fr, fr^2\}$ that perform the transformations shown in figure 1.5.

    [Answer: $g = r^2, h = rf$.]

11. If r represents rotation by 120 degrees, and f represents flipping over the axis shown, find the elements of $D_3 = \{e, r, r^2, f, fr, fr^2\}$ that perform the transformations shown in figure 1.6.

    [Answer: $g = r^2, h = f$.]

Figure 1.5: Two elements of D_3 transforming a triangle.

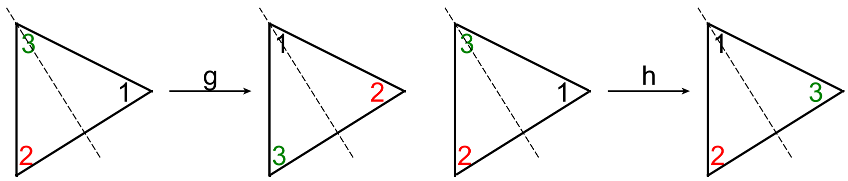
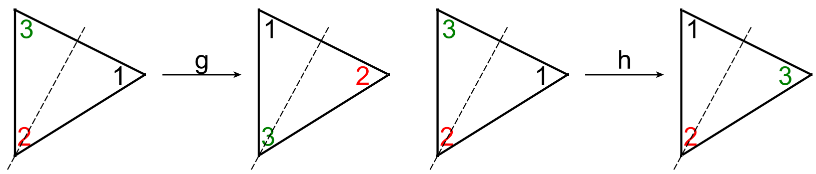


Figure 1.6: Two elements of D_3 transforming a triangle.



1.1.3 The Group D_4

1. Find the orders of the elements of D_4.
   [Answer:

element	e	r	r^2	r^3	f	rf	r^2f	r^3f
order	1	4	2	4	2	2	2	2

.]

2. How many elements are there of each order in D_4?
   [Answer:

order	1	2	3	4
elts	1	5	0	2

.]

3. Find the inverses of the elements of D_4.
   [Answer:

elt	e	r	r^2	r^3	f	rf	r^2f	r^3f
inverse	e	r^3	r^2	r	f	rf	r^2f	r^3f

.]

4. Find the center of D_4.
   [Answer: $\{e, r^2\}$]

5. Compute the following products in D_4.

   (a) $r^2f \times rf$
   [Answer: r.]

   (b) $rf \times r^2$
   [Answer: r^3f.]

   (c) $rf \times r$
   [Answer: f.]

   (d) $f \times rf$
   [Answer: r^3.]

   (e) $r^2f \times rf \times r^2f$
   [Answer: r^3f.]

(f) $rf \times r^2 \times f \times r$
[Answer: e.]

(g) $f \times rf \times r^2 f$
[Answer: rf.]

(h) $f \times r^2 f \times rf$
[Answer: $r^3 f$.]

(i) $r^3 f \times r^2 f \times r^3 f$
[Answer: f.]

6. Solve the following equations for x in D_4.

(a) $rfx = r^3$
[Answer: $r^2 f$.]

(b) $r^2 fx = rf$
[Answer: r.]

(c) $r^2 x = rf$
[Answer: $r^3 f$.]

(d) $rx = r^2 f$
[Answer: rf.]

(e) $rfx = r$
[Answer: f.]

7. Which elements do we get when we conjugate x by a for the following x and a? [Recall this means taking axa^{-1} and that this equivalent to mapping x through the inner automorphism $\phi_a(z)$.]

(a) $x = r^2, a = f$
[Answer: r^2.]

(b) $x = r, a = f$
[Answer: r^3.]

(c) $x = rf, a = f$
[Answer: $r^3 f$.]

(d) $x = r, a = rf$
[Answer: r^3.]

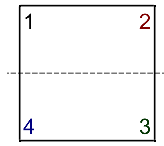
8. Make a Cayley table for the group D_4.

[Answer:

$\times$	e	r	r^2	r^3	f	rf	$r^2 f$	$r^3 f$
e	e	r	r^2	r^3	f	rf	$r^2 f$	$r^3 f$
r	r	r^2	r^3	e	rf	$r^2 f$	$r^3 f$	f
r^2	r^2	r^3	e	r	$r^2 f$	$r^3 f$	f	rf
r^3	r^3	e	r	r^2	$r^3 f$	f	rf	$r^2 f$
f	f	$r^3 f$	$r^2 f$	rf	e	r^3	r^2	r
rf	rf	f	$r^3 f$	$r^2 f$	r	e	r^3	r^2
$r^2 f$	$r^2 f$	rf	f	$r^3 f$	r^2	r	e	r^3
$r^3 f$	$r^3 f$	$r^2 f$	rf	f	r^3	r^2	r	e

.]

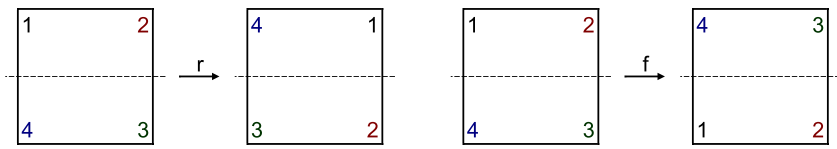
Figure 1.7: A labeled square.



9. Consider a square with its corners numbered as shown in figure 1.7.

   Here, the dotted line represents the axis for reflection. Let r be the map that rotates this square ninety degrees, and f be the map that reflects it over the axis shown. The results of these two transformations are shown in figure 1.8.

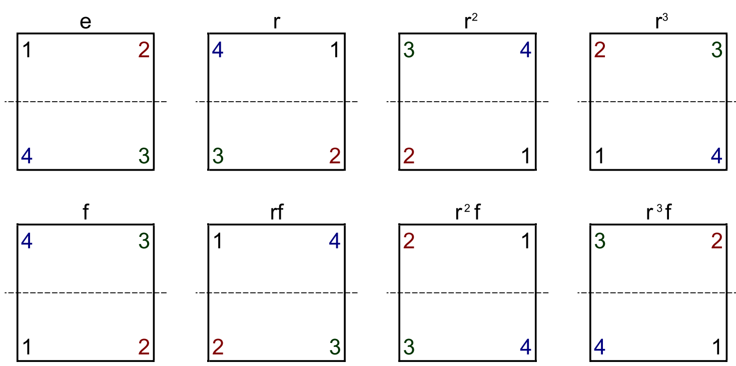
   Figure 1.8: The transformations r and f.

   For each element in D_4, make a diagram showing the position of the four corners after the motion for that element is applied.

   [Answer: See figure 1.9]

   Figure 1.9: A labeled square after individual elements of D_4 have been applied.

10. If r represents rotation by 90 degrees, and f represents flipping over the axis shown, find the elements of $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$ that perform the transformations shown in figure 1.10.

    [Answer: $g = rf, h = r^2 f$.]

11. If r represents rotation by 90 degrees, and f represents flipping over the axis shown, find the elements of $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$ that perform the transformations shown in figure 1.11.

Figure 1.10: Two elements of D_4 transforming a square.

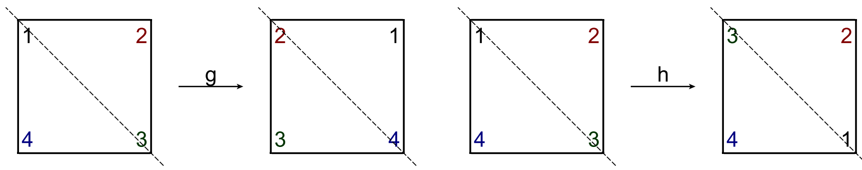
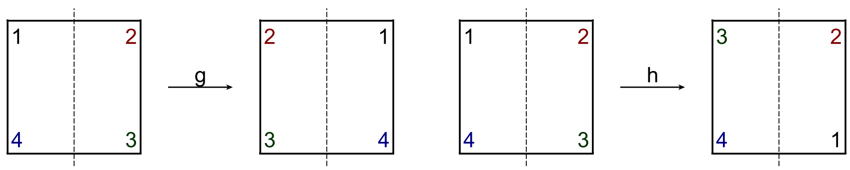


Figure 1.11: Two elements of D_4 transforming a square.



[Answer: $g = f, h = rf$.]

1.1.4 The Group D_5

1. Find the orders of the elements of D_5.

   [Answer:

element	e	r	r^2	r^3	r^4	f	rf	r^2f	r^3f	r^4f
order	1	5	5	5	5	2	2	2	2	2

   .]

2. How many elements are there of each order in D_5?

   [Answer:

order	1	2	3	4	5
elts	1	5	0	0	4

   .]

3. Find the inverses of the elements of D_5.

   [Answer:

elt	e	r	r^2	r^3	r^4	f	rf	r^2f	r^3f	r^4f
inverse	e	r^4	r^3	r^2	r	f	rf	r^2f	r^3f	r^4f

   .]

4. Find the center of D_5.
   [Answer: $\{e\}$.]

5. Compute the following products in D_5.

   (a) $rf \times r^4$
   [Answer: r^2f.]

   (b) $rf \times r$
   [Answer: f.]

   (c) $f \times rf$
   [Answer: r^4.]

   (d) $r^2f \times r^3f$
   [Answer: r^4.]

(e) $r^2f \times rf \times r^2f$
[Answer: r^3f.]

(f) $rf \times r^2 \times f \times r$
[Answer: e.]

(g) $f \times rf \times r^2f \times r^3f \times r^4f$
[Answer: r^2f.]

6. Solve the following equations for x in D_5.

(a) $rfx = r^3$
[Answer: r^3f.]

(b) $r^2fx = rf$
[Answer: r.]

(c) $r^2x = rf$
[Answer: r^4f.]

(d) $rx = r^3f$
[Answer: r^2f.]

(e) $r^4x = r^2f$
[Answer: r^3f.]

(f) $rfx = r$
[Answer: f.]

7. Which elements do we get when we conjugate x by a for the following x and a in D_5. [Recall this means taking axa^{-1} and that this equivalent to mapping x through the inner automorphism $\phi_a(z)$.]

(a) $x = r^2, a = f$
[Answer: r^3.]

(b) $x = r, a = f$
[Answer: r^4.]

(c) $x = r^3, a = r^2$
[Answer: r^3.]

(d) $x = r^3, a = f$
[Answer: r^2.]

(e) $x = r^2, a = r^3f$
[Answer: r^3.]

(f) $x = rf, a = f$
[Answer: r^4f.]

(g) $x = r, a = rf$
[Answer: r^4.]

(h) $x = r^4f, a = r^2$
[Answer: r^3f.]

(i) $x = r^4f, a = r^3$
[Answer: f.]

8. Make a Cayley table for the group D_5.

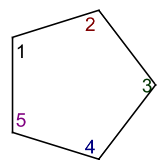
[Answer:

$\times$	e	r	r^2	r^3	r^4	f	rf	r^2f	r^3f	r^4f
e	e	r	r^2	r^3	r^4	f	rf	r^2f	r^3f	r^4f
r	r	r^2	r^3	r^4	e	rf	r^2f	r^3f	r^4f	f
r^2	r^2	r^3	r^4	e	r	r^2f	r^3f	r^4f	f	rf
r^3	r^3	r^4	e	r	r^2	r^3f	r^4f	f	rf	r^2f
r^4	r^4	e	r	r^2	r^3	r^4f	f	rf	r^2f	r^3f
f	f	r^4f	r^3f	r^2f	rf	e	r^4	r^3	r^2	r
rf	rf	f	r^4f	r^3f	r^2f	r	e	r^4	r^3	r^2
r^2f	r^2f	rf	f	r^4f	r^3f	r^2	r	e	r^4	r^3
r^3f	r^3f	r^2f	rf	f	r^4f	r^3	r^2	r	e	r^2
r^4f	r^4f	r^3f	r^2f	rf	f	r^4	r^3	r^2	r	e

.]

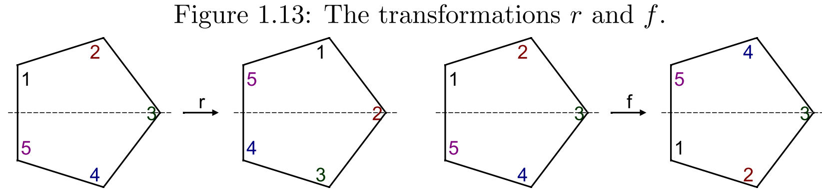
9. Consider a pentagon with its corners numbered as shown in figure 1.12.

Figure 1.12: A labeled pentagon.



Let r be the map that rotates this pentagon eighty degrees, and f be the map that reflects it over the axis shown and represented by the dotted line. The results of these two transformations are shown in figure 1.13.

Figure 1.13: The transformations r and f.



For each element in D_5, make a diagram showing the position of the corners after the motion for that element is applied.

[Answer: See figure 1.14]

10. If r represents rotation by 72 degrees, and f represents flipping over the axis shown, find the elements of $D_5 = \{e, r, r^2, r^3, r^4, f, fr, fr^2, fr^3, fr^4\}$ that perform the transformations shown in figure 1.15.

[Answer: $g = r^3, h = r^2f$.]

11. If r represents rotation by 72 degrees, and f represents flipping over the axis shown, find the elements of $D_5 = \{e, r, r^2, r^3, r^4, f, fr, fr^2, fr^3, fr^4\}$ that perform the transformations shown in figure 1.16.

[Answer: $g = rf, h = r^3$.]

Figure 1.14: A labeled square after individual elements of D_4 have been applied.

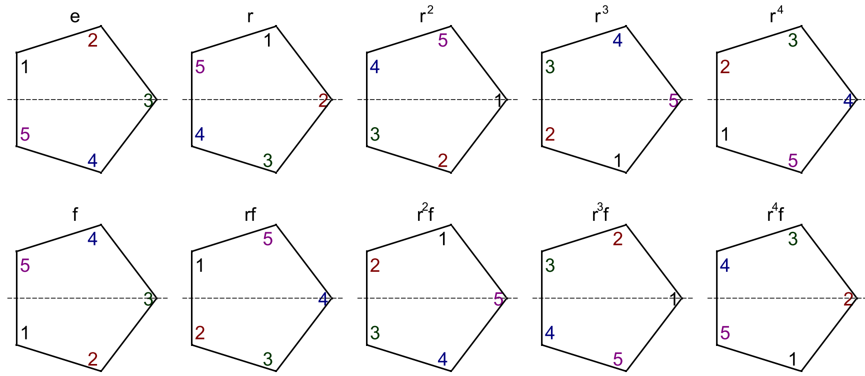


Figure 1.15: Two elements of D_5 transforming a pentagon.

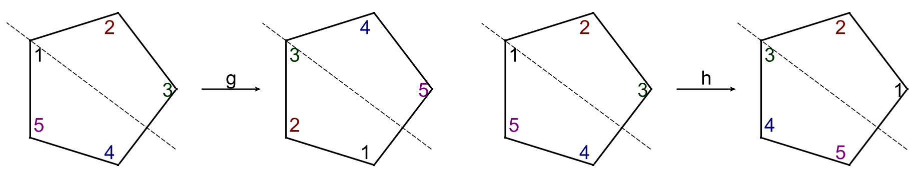
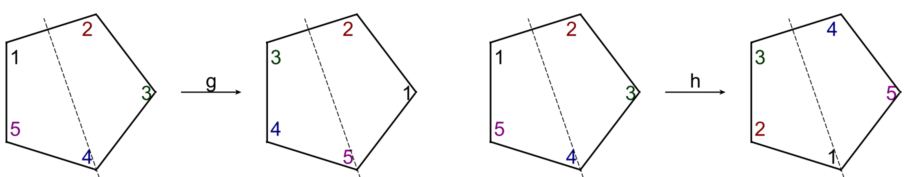


Figure 1.16: Two elements of D_5 transforming a pentagon.



1.1.5 The Group D_6

1. Find the orders of the elements of D_6.

   [Answer:

element	e	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
order	1	6	3	2	3	6	2	2	2	2	2	2

   .]

2. How many elements are there of each order in D_6?

   [Answer:

order	1	2	3	4	5	6
elts	1	7	2	0	0	2

   .]

3. Find the inverses of the elements of D_6.

   [Answer:

elt	e	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
inverse	e	r^5	r^4	r^3	r^2	r	f	rf	r^2f	r^3f	r^4f	r^5f

   .]

4. Find the center of D_6.

[Answer: $\{e, r^3\}$.]

5. Compute the following products in D_6.
   (a) $r^5 f \times rf$
   [Answer: $r^4 f$.]
   (b) $rf \times r^4$
   [Answer: $r^3 f$.]
   (c) $rf \times r$
   [Answer: f.]
   (d) $f \times rf$
   [Answer: r^5.]
   (e) $r^2 f \times rf \times r^2 f$
   [Answer: $r^3 f$.]
   (f) $rf \times r^2 \times f \times r$
   [Answer: e.]
   (g) $f \times rf \times r^2 f$
   [Answer: rf.]
   (h) $f \times r^2 f \times rf$
   [Answer: $r^5 f$.]
   (i) $r^3 f \times r^2 f \times r^3 f$
   [Answer: $r^2 f$.]

6. Solve the following equations for x in D_6.
   (a) $rfx = r^3$
   [Answer: $r^4 f$.]
   (b) $r^2 fx = rf$
   [Answer: r.]
   (c) $r^2 x = rf$
   [Answer: $r^5 f$.]
   (d) $rx = r^3 f$
   [Answer: $r^2 f$.]
   (e) $r^4 x = r^2 f$
   [Answer: $r^4 f$.]
   (f) $rfx = r$
   [Answer: f.]

7. Which elements do we get when we conjugate x by a for the following x and a in D_6. [Recall this means taking axa^{-1} and that this equivalent to mapping x through the inner automorphism $\phi_a(z)$.]
   (a) $x = r^2, a = f$
   [Answer: r^4.]

(b) $x = r, a = f$
[Answer: r^5.]

(c) $x = r^3, a = f$
[Answer: r^3.]

(d) $x = r^2, a = r^3 f$
[Answer: r^4.]

(e) $x = rf, a = f$
[Answer: $r^5 f$.]

(f) $x = r, a = rf$
[Answer: r^5.]

(g) $x = r^4 f, a = r^2 f$
[Answer: $r^2 f$.]

8. Make a Cayley table for the group D_6.

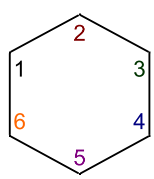
[Answer:

$\times$	e	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
e	e	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
r	r	r^2	r^3	r^4	r^5	e	rf	r^2f	r^3f	r^4f	r^5f	f
r^2	r^2	r^3	r^4	r^5	e	r	r^2f	r^3f	r^4f	r^5f	f	rf
r^3	r^3	r^4	r^5	e	r	r^2	r^3f	r^4f	r^5f	f	rf	r^2f
r^4	r^4	r^5	e	r	r^2	r^3	r^4f	r^5f	f	rf	r^2f	r^3f
r^5	r^5	e	r	r^2	r^3	r^4	r^5f	f	rf	r^2f	r^3f	r^4f
f	f	r^5f	r^4f	r^3f	r^2f	rf	e	r^5	r^4	r^3	r^2	r
rf	rf	f	r^5f	r^4f	r^3f	r^2f	r	e	r^5	r^4	r^3	r^2
r^2f	r^2f	rf	f	r^5f	r^4f	r^3f	r^2	r	e	r^5	r^4	r^3
r^3f	r^3f	r^2f	rf	f	r^5f	r^4f	r^3	r^2	r	e	r^5	r^4
r^4f	r^4f	r^3f	r^2f	rf	f	r^5f	r^4	r^3	r^2	r	e	r^5
r^5f	r^5f	r^4f	r^3f	r^2f	rf	f	r^5	r^4	r^3	r^2	r	e

.]

9. Consider a hexagon with its corners numbered as shown in figure 1.17.

Figure 1.17: A labeled hexagon.



Let r be the map that rotates this hexagon eighty degrees, and f be the map that reflects it over the axis shown and represented by the dotted line. The results of these two transformations are shown in figure 1.18.

For each element in D_6, make a diagram showing the position of the corners after the motion for that element is applied.

[Answer: See figure 1.19]

Figure 1.18: The transformations r and f.

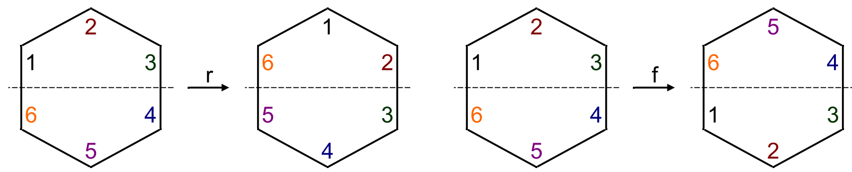
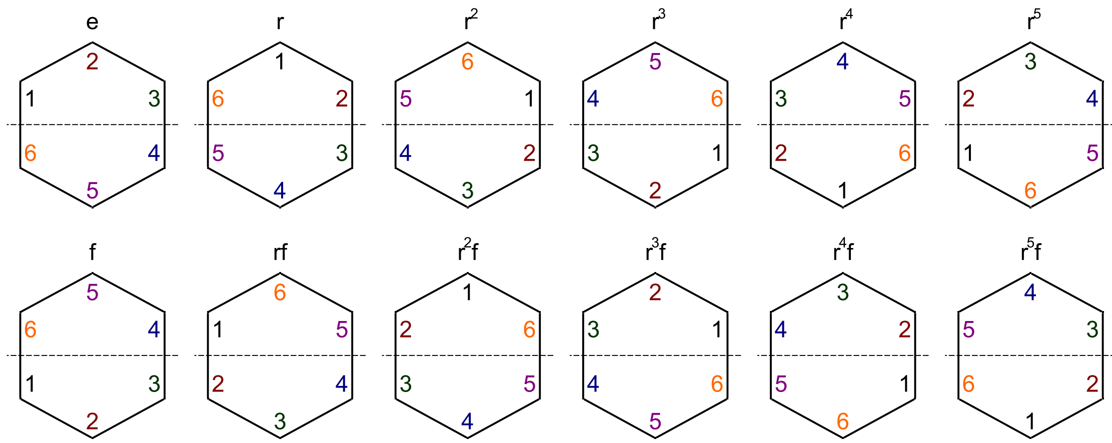
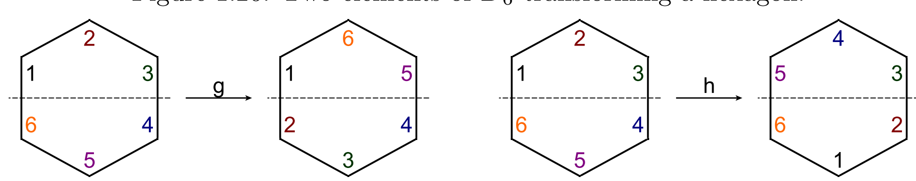


Figure 1.19: A labeled hexagon after individual elements of D_6 have been applied.



10. If r represents rotation by 60 degrees, and f represents flipping over the axis shown, find the elements of $D_6 = \{e, r, r^2, r^3, r^4, r^5, f, fr, fr^2, fr^3, fr^4, fr^5\}$ that perform the transformations shown in figure 1.20.

Figure 1.20: Two elements of D_6 transforming a hexagon.



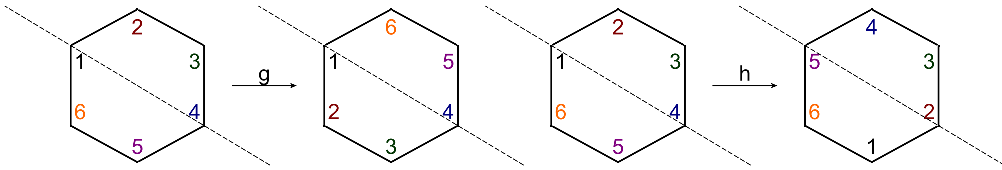
[Answer: $g = rf, h = r^5 f$.]

11. If r represents rotation by 60 degrees, and f represents flipping over the axis shown, find the elements of $D_6 = \{e, r, r^2, r^3, r^4, r^5, f, fr, fr^2, fr^3, fr^4, fr^5\}$ that perform the transformations shown in figure 1.21.

[Answer: $g = f, h = r^4 f$.]

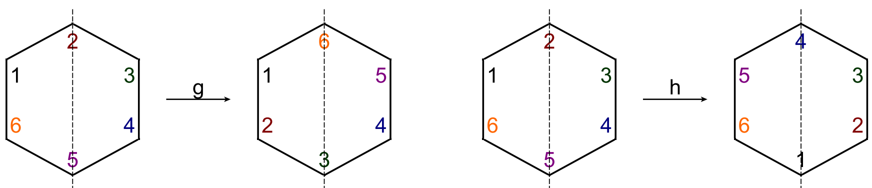
12. If r represents rotation by 60 degrees, and f represents flipping over the axis shown, find the elements

Figure 1.21: Two elements of D_6 transforming a hexagon.



of $D_6 = \{e, r, r^2, r^3, r^4, r^5, f, fr, fr^2, fr^3, fr^4, fr^5\}$ that perform the transformations shown in figure 1.22.

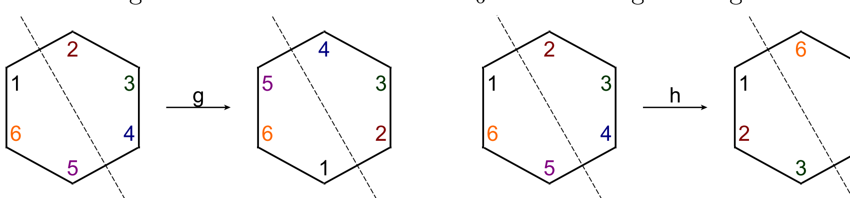
Figure 1.22: Two elements of D_6 transforming a hexagon.



[Answer: $g = r^4 f, h = r^2 f$.]

13. If r represents rotation by 60 degrees, and f represents flipping over the axis shown, find the elements of $D_6 = \{e, r, r^2, r^3, r^4, r^5, f, fr, fr^2, fr^3, fr^4, fr^5\}$ that perform the transformations shown in figure 1.23.

Figure 1.23: Two elements of D_6 transforming a hexagon.



[Answer: $g = r^3 f, h = r^5 f$.]

1.2 The Symmetric Groups

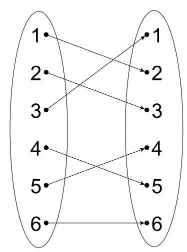
A permutation of the elements in some ordered set is a bijection from that set to itself. The set of permutations is therefore the collection of all maps from that set to itself that are both injective and surjective. If the set is finite, as it generally will be throughout this work, we only need to check one of the two properties, as injectivity implies surjectivity, and vice versa, for any finite set. As composition is associative, the identity map is a permutation, and inverses of permutations are permutations, the set of all permutations forms a group. For the set $\{1, 2, \cdots, n\}$ we call the group of all permutations the symmetric group and refer to it as S_n.

One does not have to include all possible permutations in order to form a group. Any collection of permutations containing the identity and closed under composition and inverses will form a group. If the set is finite, then it is enough to check only for closure under composition. We call such groups: permutation groups.

For one example, the map $f(1) = 2, f(2) = 3, f(3) = 1, f(4) = 5, f(5) = 4, f(6) = 6$ is a permutation of the set $\{1, 2, 3, 4, 5, 6\}$. Although this is a perfectly fine way of depicting this permutation, there are many ways of representing this same map. We can represent this permutation with a two-row matrix. The top row lists each of the elements in order, and directly below each element we list where it is sent. Thus f is represented by $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{bmatrix}$. Sometimes the top row is removed, when it is understood to be the elements in order. The same permutation could therefore be written as $\begin{bmatrix} 2 & 3 & 1 & 5 & 4 & 6 \end{bmatrix}$.

For a more visual method, if we draw a dot for each element in our set we can physically draw arrows to indicate where each element of our set is sent. The result is shown in figure 1.24. We see that the map is injective as no two arrows have the same destination, and that it is surjective as each element in the set is reached.

Figure 1.24: An arrow diagram representing the permutation where $f(1) = 2, f(2) = 3, f(3) = 1, f(4) = 5, f(5) = 4$, and $f(6) = 6$.



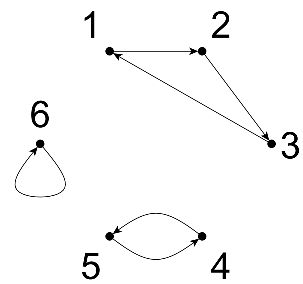
It can be considered redundant to draw both the domain and the codomain, as both sets are the same, and the arrows all point from the domain to the codomain. Instead, we can simply draw a single point for each element of the set and allow our arrows to do the rest of the work. The same permutation in figure 1.24 is redrawn in this manner in figure 1.25.

We can encapsulate the information in diagrams such as 1.25 in a very compact fashion by defining a $k-$cycle as follows: Write $(a_1, a_2, \cdots, a_k)$ whenever a_i is sent to to a_{i+1} for $i < k$, and a_k to a_1. Any permutation can be written as a combination k-cycles, each containing no elements in common. For example, the permutation depicted in figures 1.24 and 1.25 becomes $(1, 2, 3)(4, 5)(6)$. Note that such a representation is not unique, as we could have also written $(2, 3, 1)(4, 5)(6)$, $(5, 4)(6)(3, 1, 2)$, or any of a large number of possibilities[1].

Just as the fraction $\frac{2}{4}$ equals the fraction $\frac{3}{6}$, there is nothing wrong with having more than one way to represent the same element. However, we can reduce the number of possibilities by writing each individual cycle starting with the smallest element in our ordering. We can choose to leave out $1-$cycles understanding that any element not appearing is automatically sent to itself, and we can use the letter e to denote the permutation where every element is sent to itself. Finally, if we then list the individual cycles in lexicographic

[1]There are thirty-six ways to write this one permutation so that the cycle (6) appears, in fact.

Figure 1.25: An different arrow diagram representing the permutation where $f(1) = 2, f(2) = 3, f(3) = 1, f(4) = 5, f(5) = 4$, and $f(6) = 6$.



order, each group element can be written in only one way. Permutations written in this way will be said to be in *reduced form*. For example the cycle $(8,5)(2,1,4,7)(6)(3)$ can only be written in reduced form as $(2,4,7,2)(5,8)$.

As with other groups, we often use juxaposition to denote multiplication. Thus in S_6, $(1,4)(2,5,6)$ could denote either the product $(1,4) \times (2,5,6)$ or the element $(1,4)(2,5,6)$. This is fine, because in this and all possible cases, the two are the same. If we wrote $(1,4)(2,4,6)$ then this must represent the product $(1,4) \times (2,4,6)$ since $(1,4)(2,4,6)$ does not represent anything in cycle notation. There are many ways of writing a permutation in cycle notation without resorting to the reduced form, but in all of them, the elements of each individual cycle must be disjoint.

There are many advantages to writing permutations in cycle notation. Inverses can be computed simply by reversing the order of the elements in each cycle. For example, the inverse of $(1,2,3)(4,5)$ is simply $(3,2,1),(5,4)$. The order of a permutation is the least common multiple of the lengths of the cycles. Thus we can immediately see the order of $(1,2,3)(4,5)$ is $LCM(3,2) = 6$.

We can write any element in cycle notation as a product of 2-cycles, by using the formula $(a_1, a_2, \cdots, a_n) = (a_1, a_n) \times (a_1, a_{n-1}) \times \cdots \times (a_1, a_2)$. Applying this to the permutation $(1,2,3)(4,5,6,7)$ in S_7 gives us the product $(1,3) \times (1,2) \times (4,7) \times (4,6) \times (4,5)$. The outcome is not itself in cycle notation, because the 2-cycles will not generally be disjoint, though this is still useful. One of the most important results of a first course in group theory is that every element can be written as either an even number or an odd number of 2-cycles, but none can be written as both. We call the former *even permutations* and the rest *odd permutations*. The set of all even permutations in S_n is called the alternating group A_n.

1.2.1 The Group S_3

1. Write the following elements of S_3 in reduced form.

   (a) $(3,1,2)$
   [Answer: $(1,2,3)$]

   (b) $(1)(3,2)$
   [Answer: $(2,3)$]

   (c) $(3,2)(1)$
   [Answer: $(2,3)$]

(d) $(3,2)$
[Answer: $(2,3)$]

(e) $(1)(2,3)$
[Answer: $(2,3)$]

2. Compute the following products in S_3.

(a) $(1,3,2)\times(1,3,2)$
[Answer: $(1,2,3)$]

(b) $(1,3,2)\times(1,2)$
[Answer: $(2,3)$]

(c) $(1,2)\times(1,3,2)$
[Answer: $(1,3)$]

(d) $(1,2)\times(1,3)$
[Answer: $(1,3,2)$]

(e) $(1,2)\times(2,3)$
[Answer: $(1,2,3)$]

3. Find the order of the following elements in S_3.

(a) e
[Answer: 1]

(b) $(1,2)$
[Answer: 2]

(c) $(1,2,3)$
[Answer: 3]

4. Find the inverse of the following elements in S_3.

(a) e
[Answer: e]

(b) $(1,2)$
[Answer: $(1,2)$]

(c) $(1,2,3)$
[Answer: $(1,3,2)$]

5. Solve the following equations in S_3.

(a) $(1,2)x=(2,3))$
[Answer: $(1,2,3)$]

(b) $(1,2)x=(1,3))$
[Answer: $(1,3,2)$]

(c) $(1,2,3)x=(1,2)$
[Answer: $(2,3)$]

(d) $(1,3,2)x = (1,2)$
[Answer: $(1,3)$]

(e) $(1,2)x = (1,2,3)$
[Answer: $(2,3)$]

(f) $(1,2)x = (1,3,2)$
[Answer: $(1,3)$]

(g) $(1,2,3)x = (1,3,2)$
[Answer: $(1,2,3)$]

(h) $(1,2,3)x = (1,2,3)$
[Answer: e]

6. Conjugate the element a by the element b in the following cases.

(a) $a = (1,2,3), b = (1,2)$.
[Answer: $(1,3,2)$]

(b) $a = (1,2,3), b = (1,3)$.
[Answer: $(1,3,2)$]

(c) $a = (1,2,3), b = (2,3)$.
[Answer: $(1,3,2)$]

(d) $a = (1,2), b = (1,2,3)$.
[Answer: $(1,3)$]

(e) $a = (1,2), b = (1,3,2)$.
[Answer: $(2,3)$]

(f) $a = (1,2), b = (2,3)$.
[Answer: $(1,3)$]

(g) $a = (1,2), b = (1,3)$.
[Answer: $(2,3)$]

7. Write each of the following elements as a product of 2-cycles and then state whether it is odd or even. There is more than one answer to the first part of the following questions.

(a) $(1,2,3)$
[Answer: $(1,3)(1,2)$. Even.]

(b) $(1,3,2)$
[Answer: $(1,2)(1,3)$. Even.]

(c) $(1,3)$
[Answer: $(1,3)$. Odd.]

8. How many 2-cycles are in S_3
[Answer: We can pick two elements in $\binom{3}{2}$ ways. Since the cycle (a,b) is the same as the cycle (b,a), each of these gives us only one element. Thus there are exactly three 2-cycles.]

9. How many 3-cycles are in S_3?
[Answer: There are 3! ways to arrange the three numbers. Noting that $(1,2,3) = (2,3,1) = (3,1,2)$ and $(1,3,2) = (3,2,1) = (2,1,3)$ shows there are only two distinct 3-cycles.]

10. How many elements are in S_3?
[Answer: Adding the 2-cycles, the 3-cycles and the identity gives us $2+3+1=6$. We could also have just taken 3!.]

11. How many even permutations are there in S_3?
[Answer: The 2-cycles are automatically an odd number of 2-cycles and that number is one. A 3-cycle (a,b,c) is $(a,c)(a,b)$ and thus even. The identity is also even. Thus we get 3 total even permutations. We could have also used that exactly half the permutations are even to get that there are $\frac{6}{2} = 3$ even permutations.

12. Find the inverse of each element of S_3.
[Answer: A 2-cycle is its own inverse. The inverse of (1,2,3) is (1,3,2) and vice versa. Our remaining element is the identity which is also its own inverse.]

13. How many subgroups does S_3 have?
[Answer: Every proper subgroup must have order 1,2,3 or 6. We know that $\{e\}$ and S_3 are subgroups. Each of the 2-cycles (a,b) generates a distinct subgroup $\{e,(a,b)\}$. We cannot add an element g to these without including both g and $(a,b)g$. This forces us past size three and makes us include all six elements. Either three cycle generates the subgroup $\{e,(1,2,3),(1,3,2)\}$. Any subgroup containing this would have to have order 6 and thus be all of S_3. Thus we get six subgroups of S_3.]

14. Prove that every proper subgroup is S_3 is abelian.
[Answer: The last argument showed that each proper subgroup was cyclic, and hence is abelian.]

15. If we multiply a 2-cycle and a 3-cycles in S_3 is the outcome always a 2-cycle, always a 3-cycle, or always the identity?
[Answer: The product must be odd, so it must be a 2-cycle.]

16. If we multiply a 2-cycle and a 2-cycles in S_3 is the outcome always a 2-cycle, always a 3-cycle, or always the identity?
[Answer: The outcome must be even, so it can be either a 3-cycle or the identity. In fact, both are possible. If the two 2-cycles are the same we get the identity. Otherwise we get a 3-cycle.]

1.2.2 The Group S_4

1. Write the following elements of S_4 in reduced form.

   (a) $(3,1,2)(4)$
   [Answer: $(1,2,3)$]

   (b) $(4,2)(3,1)$
   [Answer: $(1,3)(2,4)$]

   (c) $(3,2)(1)(4)$
   [Answer: $(2,3)$]

2. Compute the following products in S_4.

   (a) $(1,2,3) \times (1,2)(3,4)$
   [Answer: $(1,3,4)$]

   (b) $(1,3,2) \times (1,2)(3,4)$
   [Answer: $(2,3,4)$]

   (c) $(1,2)(3,4) \times (1,3,2)$
   [Answer: $(1,4,3)$]

   (d) $(1,2)(3,4) \times (1,3)(2,4)$
   [Answer: $(1,4)(2,3)$]

   (e) $(1,2,3,4) \times (1,3)(2,4)$
   [Answer: $(1,4,3,2)$]

   (f) $(1,2,3,4) \times (1,2,3)$
   [Answer: $(1,3,2,4)$]

3. Find the order of the following elements in S_4.

   (a) $(4,3)$
   [Answer: 2]

   (b) $(4,3,2)$
   [Answer: 3]

   (c) $(1,4,3,2)$
   [Answer: 4]

   (d) $(1,3)(2,4)$
   [Answer: 2]

4. Find the inverse of the following elements in S_4.

   (a) $(3,4)$
   [Answer: $(3,4)$]

   (b) $(2,4,3)$
   [Answer: $(2,3,4)$]

   (c) $(1,4,3,2)$
   [Answer: $(1,2,3,4)$]

   (d) $(1,3)(2,4)$
   [Answer: $(1,3)(2,4)$]

5. Solve the following equations in S_4.

   (a) $(1,2,3)x = (1,2)(3,4)$
   [Answer: $(2,3,4)$]

   (b) $(1,2,3)x = (1,2)$
   [Answer: $(2,3)$]

(c) $(1,3,2)x = (1,2)$
[Answer: $(1,3)$]

(d) $(1,2)(3,4)x = (1,3,2)$
[Answer: $(1,4,3)$]

(e) $(1,2)(3,4)x = (1,3)(2,4)$
[Answer: $(1,4)(2,3)$]

(f) $(1,2,3,4)x = (1,3)(2,4)$
[Answer: $(1,2,3,4)$]

(g) $(1,2,3,4)x = (1,2,3)$
[Answer: $(3,4)$]

6. Conjugate the element a by the element b in the following cases.

   (a) $a = (1,2,3), b = (1,2)(3,4)$.
   [Answer: As $b = b^{-1}$ for this b, we get $(1,2)(3,4)(1,2,3)(1,2)(3,4) = (1,4,2)$]

   (b) $a = (1,2)(3,4), b = (1,2,3)$.
   [Answer: $(2,3)(1,4)$]

   (c) $a = (1,2,3), b = (1,2)$.
   [Answer: $(1,3,2)$]

   (d) $a = (1,2), b = (1,3,2)$.
   [Answer: $(1,3)$]

   (e) $a = (1,3,2,4), b = (1,4,3,2)$.
   [Answer: $(1,3,4,2)$] 1432 1324 1234

   (f) $a = (1,4)(3,2), b = (1,3,2,4)$.
   [Answer: $(1,2)(3,4)$] 1432 14 32 1234

7. Write each of the following elements as a product of 2-cycles and then state whether it is odd or even. There is more than one answer to the first part of the following questions.

   (a) $(1,2,3,4)$
   [Answer: $(1,4)(1,3)(1,2)$. Odd.]

   (b) $(1,3,4)$
   [Answer: $(1,4)(1,3)$. Even.]

   (c) $(1,3)(2,4)$
   [Answer: $(1,3)(2,4)$. Even.]

8. How many 2-cycles are there in S_4?
[Answer: There are $\binom{4}{2} = 6$ ways to pick two elements, and since $(a,b) = (b,a)$ each distinct pair generates one 2-cycle. Thus we get six 2-cycles.]

9. List the elements that are 2-cycles in S_4. [Answer: $(1,2),(1,3),(1,4)(2,3),(2,4)$, and $(3,4)$.]

10. How many 3-cycles are there in S_4?
[Answer: There are $\binom{4}{3} = 4$ ways to pick three elements from the underlying set, and any three elements make two 3-cycles $(a, b, c) = (b, c, a) = (c, a, b)$ and $(a, c, b) = (c, b, a) = (b, a, c)$. Thus we get $4 \times 2 = 8$ different 3-cycles.]

11. List the elements that are 3-cycles in S_4. [Answer: $(1, 2, 3), (1, 3, 2), (1, 2, 4), (1, 4, 2), (1, 3, 4), (1, 4, 3), (2, 3, 4)$, and $(2, 4, 3)$.]

12. How many pairs of disjoint 2-cycles are there in S_4?
[Answer: Once we pick one 2-cycle, then the other is determined. Picking either of these determines the pair. Thus we get 6/2=3 pairs of disjoint 2-cycles.]

13. List the elements that are pairs of disjoint 2-cycles in S_4. [Answer: $(1, 2)(3, 4), (1, 3)(2, 4)$, and $(1, 4)(2, 3)$.]

14. How many 4-cycles are there in S_4?
[Answer: Any 4-cycle can be written with the "1" listed first, and then there are 3!=6 ways to list the other elements next, each resulting in a different 4-cycle. This gives us six 4-cycles.]

15. List the elements that are 4-cycles in S_4. [Answer: $(1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 2, 4), (1, 3, 4, 2), (1, 4, 2, 3)$, and $(1, 4, 3, 2)$.]

16. How many elements are there in S_4?
[Answer: We can add up the previous answers tossing in 1 for the identity to get 6+8+3+6+1=24 or simply take $4! = 24$.]

17. How many even permutations are there in S_4?
[Answer: We can add up the even permutations from the problems above, which would be the 3-cycles, and pairs of disjoint 2-cycles together with the identity. This gives us 8+3+1=12. Or we can use the result that says half of the permutations will always be even to get 24/2 which is also 12.]

18. What are the inverses of each of the elements in S_4?
[Answer: The inverse of any 2-cycle or pair of disjoint 2-cycles is itself. The identity is always its own inverse. Thus we have left to consider the 3-cycles and 4-cycles. Here in S_4 (a, b, c) will have inverse (a, c, b) and (a, b, c, d) will have inverse (a, d, c, b) for any distinct elements a, b, c, d in $\{1, 2, 3, 4\}$.]

19. How many elements are there of each possible order in S_4?
[Answer: An n-cycle has order n, and a product of disjoint n-cycles has order of the least common multiple of the orders of these n-cycles. Thus 2-cycles and pairs of disjoint 2-cycles have order 2, the 3-cycles have order 3, 4-cycles have order 4, and the identity has order one. This gives us the following table.

order	1	2	3	4
#elts	1	9	8	6

.]

20. How many even elements are there of each possible order in S_4?
[Answer: These are the elements of A_4. We simply remove the values for odd permutations to get the following table:

order	1	2	3
#elts	1	3	8

.]

21. Find all cyclic subgroups of order three in S_4.
[Answer: Any element of order three will generate a cyclic subgroup. We get that $<(1,2,3)>=<(1,3,2)>=\{e,(1,2,3),(1,3,2)\}$, $<(1,2,4)>=<(1,4,2)>=\{e,(1,2,4),(1,4,2)\}$, $<(1,3,4)>=<(1,4,3)>=\{e,(1,3,4),(1,4,3)\}$ and $<(2,3,4)>=<(2,4,3)>=\{e,(2,3,4),(2,4,3)\}$. Thus we have found four such subgroups. There can be no others. Any cyclic subgroup of order three needs a generator of order three, and we've already used up all eight elements of order three.]

1.2.3 The Group S_5

1. Write the following elements of S_5 in reduced form.
   (a) $(3,1,2)(4)(5)$
   [Answer: $(1,2,3)$]
   (b) $(4,2,5)(3,1)$
   [Answer: $(1,3)(2,4,5)$]
   (c) $(4,1,5)(3,2)$
   [Answer: $(1,5,4)(2,3)$]
   (d) $(2,3)(5,4,1)$
   [Answer: $(1,5,4)(2,3)$]

2. Compute the following products in S_5.
   (a) $(1,2)\times(1,3,4,5)$
   [Answer: $(1,3,4,5,2)$]
   (b) $(1,2,3)(4,5)\times(1,2)(3,4)$
   [Answer: $(1,3,5,4)$]
   (c) $(1,2)(3,4)\times(1,2,3)(4,5)$
   [Answer: $(2,4,5,3)$]
   (d) $(1,3,2)(4,5)\times(1,2)(3,4)$
   [Answer: $(2,3,5,4)$]
   (e) $(1,3,2)(4,5)\times(1,2)(4,5)$
   [Answer: $(2,3)$]
   (f) $(1,2)(3,4)\times(1,2)(4,5)$
   [Answer: $(3,4,5)$]
   (g) $(1,2)(3,4)\times(2,3)(4,5)$
   [Answer: $(1,2,4,5,3)$]
   (h) $(1,2)(3,4)\times(3,5)(2,4)$
   [Answer: $(1,2,3,5,4)$]
   (i) $(1,2,3,4,5)\times(1,2)$
   [Answer: $(1,3,4,5)$]
   (j) $(1,2)\times(1,2,3,4,5)$
   [Answer: $(2,3,4,5)$]

(k) $(1,2,3,4,5) \times (1,3)$
[Answer: $(1,4,5)(2,3)$]

(l) $(1,2,3,4,5) \times (1,2,3,4)$
[Answer: $(1,3,5)(2,4)$]

3. Find the order of the following elements in S_5.

(a) $(1,4,3,2)$
[Answer: 4]

(b) $(1,3,2)(4,5)$
[Answer: 6]

(c) $(1,2)(3,4)$
[Answer: 2]

(d) $(1,2,4,5,3)$
[Answer: 5]

4. Find the inverse of the following elements in S_5.

(a) $(1,4,3,2)$
[Answer: $(1,2,3,4)$]

(b) $(1,3,2)(4,5)$
[Answer: $(1,2,3)(4,5)$]

(c) $(1,4)(3,5)$
[Answer: $(1,4)(3,5)$]

(d) $(1,2,4,5,3)$
[Answer: $(1,3,5,4,2)$]

5. Solve the following equations in S_5.

(a) $(1,2,3,4,5)x = (1,2,3,5,4)$
[Answer: $(3,4,5)$]

(b) $(1,2,3,4,5)x = (1,2,4,3,5)$
[Answer: $(2,3,4)$]

(c) $(1,2,3,4,5)x = (1,5,3,2,4)$
[Answer: $(1,4,5,2,3)$]

(d) $(1,2,3)x = (4,5)$
[Answer: $(1,3,2)(4,5)$]

(e) $(1,2)x = (3,4,5)$
[Answer: $(1,2)(3,4,5)$]

(f) $(1,2)(3,4)x = (2,3,4,5)$
[Answer: $(1,2,4,5)$]

(g) $(2,3,4,5)x = (1,2)(3,4)$
[Answer: $(1,5,4,2)$]

(h) $(1,2)(3,4,5)x = (1,3)(2,4,5)$
[Answer: $(1,5)(2,3)$]

(i) $(1,2)(3,4,5)x = (3,4)(1,2,5)$
[Answer: $(2,4,5)$]

(j) $(1,2,3,4)x = (1,5)$
[Answer: $(1,5,2,3,4)$]

(k) $(1,2,3,4)x = (1,2,4)(3,5)$
[Answer: $(2,3,5)$]

6. Conjugate the element a by the element b in the following cases.

(a) $a = (1,2,3,4,5), b = (1,2)(3,4,5)$.
[Answer: $(1,4,5,3,2)$]

(b) $a = (1,2,3,4,5), b = (1,3)(2,5,4)$.
[Answer: $(1,4,3,2)$]

(c) $a = (1,2,5)(3,4), b = (1,2,3)(4,5)$.
[Answer: $(1,3)(2,5,4)$]

(d) $a = (1,5,3), b = (1,2,3,4,5)$.
[Answer: $(1,4,3,5,2)$]

(e) $a = (1,2,3,4,5), b = (1,3,2)$.
[Answer: $(2,4,3)$]

(f) $a = (1,2,3,4,5), b = (1,3,2)(4,5)$.
[Answer: $(1,5)(2,4,3)$]

(g) $a = (1,2,3,4), b = (1,3,2,5)$.
[Answer: $(2,4,3,5)$]

(h) $a = (1,2,3,4), b = (1,3,4,5)$.
[Answer: $(1,5,2,4)$]

7. Write each of the following elements as a product of 2-cycles and then state whether it is odd or even. There is more than one answer to the first part of the following questions.

(a) $(1,2,3,4,5)$
[Answer: $(1,5)(1,4)(1,3)(1,2)$. Even.]

(b) $(1,5,3,2)$
[Answer: $(1,2)(1,3)(1,5)$. Odd.]

(c) $(1,5,2)$
[Answer: $(1,2)(1,5)$. Even.]

(d) $(1,3)(2,5)$
[Answer: $(1,3)(2,5)$. Even.]

(e) $(1,3,4)(2,5)$
[Answer: $(1,4)(1,3)(2,5)$. Odd.]

8. How many 2-cycles are in S_5?
[Answer: There are $\binom{5}{2} = 10$ ways to pick two elements and each gives rise to only one 2-cycle. Thus the answer is ten.]

9. List the elements that are 2-cycles in S_5. [Answer: $(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)$, and $(4,5)$.]

10. How many 3-cycles are in S_5?
[Answer: There are $\binom{5}{3} = 10$ ways to pick three elements and each of those gives rise to exactly two 3-cycles. This is because for elements a, b and c in $\{1,2,3,4,5\}$ we get $(a,b,c) = (b,c,a) = (c,a,b)$ and its inverse $(a,c,b) = (c,b,a) = (b,a,c)$. Thus the answer is twenty.]

11. List the elements that are 3-cycles in S_5. [Answer: $(1,2,3)$, $(1,3,2)$, $(1,2,4)$, $(1,4,2)$, $(1,2,5)$, $(1,5,2)$, $(1,3,4)$, $(1,4,3)$, $(1,3,5)$, $(1,5,3)$, $(1,4,5)$, $(1,5,4)$, $(2,3,4)$, $(2,4,3)$, $(2,3,5)$, $(2,5,3)$, $(2,4,5)$, $(2,5,4)$, $(3,4,5)$, and $(3,5,4)$.]

12. How many 4-cycles are in S_5?
[Answer: There are $\binom{5}{4} = 5$ ways to pick four elements and each of those gives rise to exactly six 4-cycles. To see that, note that we can always write the smallest element at the start of a cycle, and then each of the 3! possibilities for ordering the other three elements gives us something distinct. Thus the answer is thirty.]

13. List the elements that are 4-cycles in S_4. [Answer: $(1,2,3,4)$, $(1,2,4,3)$, $(1,3,2,4)$, $(1,3,4,2)$, $(1,4,2,3)$, $(1,4,3,2)$, $(1,2,3,5)$, $(1,2,5,3)$, $(1,3,2,5)$, $(1,3,5,2)$, $(1,5,2,3)$, $(1,5,3,2)$, $(1,2,5,4)$, $(1,2,4,5)$, $(1,5,2,4)$, $(1,5,4,2)$, $(1,4,2,5)$, $(1,4,5,2)$, $(1,4,3,5)$, $(1,4,5,3)$, $(1,3,4,5)$, $(1,3,5,4)$, $(1,5,4,3)$, $(1,5,3,4)$, $(2,4,3,5)$, $(2,4,5,3)$, $(2,3,4,5)$, $(2,3,5,4)$, and $(2,5,4,3)$.]

14. How many 5-cycles are in S_5?
[Answer: We can write any 5-cycle by listing the number one first and then there are 4!=24 ways to list the other elements, each which gives rise to a distinct 5-cycle. Thus we get twenty-four 5-cycles.]

15. List the elements that are 5-cycles in S_5. [Answer: $(1,2,3,4,5)$, $(1,2,3,5,4)$, $(1,2,4,3,5)$, $(1,2,4,5,3)$, $(1,2,5,3,4)$, $(1,2,5,4,3)$, $(1,3,2,4,5)$, $(1,3,2,5,4)$, $(1,3,4,2,5)$, $(1,3,4,5,2)$, $(1,3,5,2,4)$, $(1,3,5,4,2)$, $(1,4,2,3,5)$, $(1,4,2,5,3)$, $(1,4,3,2,5)$, $(1,4,3,5,2)$, $(1,4,5,3,2)$, $(1,4,5,2,3)$, $(1,5,2,3,4)$, $(1,5,2,4,3)$, $(1,5,3,2,4)$, $(1,5,3,4,2)$, $(1,5,4,3,2)$, and $(1,5,4,2,3)$.]

16. How many pairs of disjoint 2-cycles are in S_5?
[Answer: We can count this in many ways. We can first pick the element not in either 2-cycles in 5 ways, and then pick one 2-cycle in $\binom{4}{2}$ ways. The other 2-cycle is then determined. However, as we could have picked that other 2-cycle first, we have counted each possibility exactly twice. Thus we get $\frac{5\times 6}{2} = 15$ possibilities.]

17. List the elements that are pairs of disjoint 2-cycles in S_5. [Answer: $(1,2)(3,4)$, $(1,2)(3,5)$, $(1,2)(4,5)$, $(1,3)(2,4)$, $(1,3)(2,5)$, $(1,3)(4,5)$, $(1,4)(2,3)$, $(1,4)(2,5)$, $(1,4)(3,5)$, $(1,5)(2,3)$, $(1,5)(2,4)$, $(1,5)(3,4)$, $(2,3)(4,5)$ $(2,4)(3,5)$ and $(2,5)(3,4)$.]

18. How many disjoint 2-cycle and 3-cycle pairs are in S_5?
[Answer: We can count this in many ways. We can pick the elements of the 2-cycle in $\binom{5}{2}$ ways and then the remaining three elements can make two possible 3-cycles depending on the order we arrange them. Thus there are twenty possibilities.]

19. List the elements that are disjoint 2-cycle and 3-cycles pairs in S_5. [Answer: $(1,2,3)(4,5)$, $(1,3,2)(4,5)$, $(1,2,4)(3,5)$, $(1,4,2)(3,5)$, $(1,2,5)(3,4)$, $(1,5,2)(3,4)$, $(1,3,4)(2,5)$, $(1,4,3)(2,5)$, $(1,3,5)(2,4)$, $(1,5,3)(2,4)$, $(1,4,5)(2,3)$, $(1,5,4)(2,3)$, $(2,3,4)(1,5)$, $(2,4,3)(1,5)$, $(2,3,5)(1,4)$, $(2,5,3)(1,4)$, $(2,4,5)(1,3)$, $(2,5,4)(1,3)$, $(3,4,5)(1,2)$, and $(3,5,4)(1,2)$.]

20. How many elements are there in S_5?
[Answer: We can add up the total number of each of these types, with 1 for the identity to get $1+10+20+30+24+15+20=120$ or simply take $5!$.]

21. Which orders occur in S_5?
[Answer: Once an element is written in disjoint cycles, we find the order by taking the least common multiple of the lengths of those disjoint cycles. The possibilites for the order of single cycle elements are 5,4,3,2, and 1. The possibilities for 2 disjoint cycles come from a 2-cycle and a 3-cycle, which give us an element of order six, or two 2-cycles which would have order 2. Thus the list of all possible orders is 1,2,3,4,5, and 6.]

22. How many elements are there of each possible order in S_5?
[Answer: Putting together our counts of different types of elements, together with the orders from the last problem, we get the following table:

order	1	2	3	4	5	6
#elts	1	25	20	30	24	20

.]

23. How many even elements are there of each possible order in S_5? Note that this is equivalent to asking how many elements there are of each possible order in A_5.
[Answer:

order	1	2	3	5
#elts	1	15	20	24

.]

24. Does S_5 have a subgroup isomorphic to D_3?
[Answer: Yes. The permutations fixing four and five give us a subgroup isomorphic to S_3. Since S_3 is isomorphic to D_3 the answer is yes.]

25. Does S_5 have a subgroup isomorphic to the Klein-four group V?
[Answer: Yes. The group $\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is one such example.]

1.2.4 Dihedral Groups as Subgroups of Symmetric Groups

1. Let H be the subgroup of S_3 generated by the elements $r=(1,2,3)$ and $f=(1,3)$.

   (a) Show that $r^3=e$. [Answer: The order of an $n-$cycle is n so $r^3=(1,2,3)^3=e$.]

   (b) Show that $fr=r^2f$. [Answer: $fr=(1,3)(1,2,3)=(1,2)$ and $r^2f=(1,2,3)(1,2,3)(1,3)=(1,2)$ as well.]

   (c) Show that $fr^k=r^{3-k}f$ for $k\in\{1,2\}$. [Answer: This has now been shown for $k=1$ so we only need show the $k=2$ case. $fr^2=(1,3)(1,2,3)(1,2,3)=(2,3)$ and $rf=(1,2,3)(1,3)=(2,3)$.]

(d) Write the elements $e, r, r^2, f, r, r^2 f$ in cycle notation.

Elt	e	r	r^2	f	rf	$r^2 f$
Cycle	e	$(1,2,3)$	$(1,3,2)$	$(1,3)$	$(2,3)$	$(1,2)$

.]

(e) Show that the set $\{e, r, r^2, f, rf, r^2 f\}$, defined with the cycles listed above, is closed under composition. [Answer: We have seen this set equals all of S_3, which is a group, hence closed under multiplication.]

(f) Make a Cayley table using $\{e, r, r^2, f, rf, r^2 f\}$. Write each element in cycle notation.

[Answer:

$\times$	e	$(1,2,3)$	$(1,3,2)$	$(1,3)$	$(2,3)$	$(1,2)$
e	e	$(1,2,3)$	$(1,3,2)$	$(1,3)$	$(2,3)$	$(1,2)$
$(1,2,3)$	$(1,2,3)$	$(1,3,2)$	e	$(2,3)$	$(1,2)$	$(1,3)$
$(1,3,2)$	$(1,3,2)$	e	$(1,2,3)$	$(1,2)$	$(1,3)$	$(2,3)$
$(1,3)$	$(1,3)$	$(1,2)$	$(2,3)$	e	$(1,3,2)$	$(1,2,3)$
$(2,3)$	$(2,3)$	$(1,3)$	$(1,2)$	$(1,2,3)$	e	$(1,3,2)$
$(1,2)$	$(1,2)$	$(2,3)$	$(1,3)$	$(1,3,2)$	$(1,2,3)$	e

.]

2. Let H be the subgroup of S_4 generated by the elements $r = (1,2,3,4)$ and $f = (1,4)(2,3)$.

(a) Show that $r^4 = e$. [Answer: The order of an $n-$cycle is n so $r^4 = (1,2,3,4)^4 = e$.]

(b) Show that $fr = r^3 f$. [Answer: $fr = (1,4)(2,3)(1,2,3,4) = (1,3)$ and $r^3 f = (1,2,3,4)^3(1,4)(2,3) = (1,2,3,4)^{-1}(1,4)(2,3) = (4,3,2,1)(1,4)(2,3) = (1,3)$ as well.]

(c) Show that $fr^k = r^{4-k} f$ for $k \in \{0,1,2,3\}$. [Answer: The $k = 0$ case reduces to $f = f$ and this has now been shown for $k = 1$, so we only need show the $k = 2$ and $k = 3$ cases. For $k = 2$ we must show r^2 commutes with f. Noting $r^2 = (1,3)(2,4)$, this is true because $r^2 f = (1,3)(2,4)(1,4)(2,3)$ and $fr^2 = (1,4)(2,3)(1,3)(2,4)$ are both equal to $(1,2)(3,4)$. For $k = 3$ we must show $fr^3 = rf$. It is probably fastest to multiply out the cycles again, though we can also use the three facts we've already shown together. As $r^4 = e$, $fr^2 = r^2 f$, and $fr = r^3 f$, we see $rf^3 = r^2 fr = r^2 r^3 f = r^5 f = rf$.

(d) Write the elements $e, r, r^2, r^3, f, rf, r^2 f, r^3 f$ in cycle notation. [Answer:

Elt	e	r	r^2	r^3	f	rf	$r^2 f$	$r^3 f$
Cycle	e	$(1,2,3,4)$	$(1,3)(2,4)$	$(1,4,3,2)$	$(1,4)(2,3)$	$(2,4)$	$(1,2)(3,4)$	$(1,3)$

.]

(e) Make a Cayley table using the set $\{e, r, r^2, r^3, f, rf, r^2 f, r^3 f\}$. Write each element in cycle notation. [Answer:

$\times$	e	$(1,2,3,4)$	$(1,3)(2,4)$	$(1,4,3,2)$	$(1,4)(2,3)$	$(2,4)$	$(1,2)(3,4)$	$(1,3)$
e	e	$(1,2,3,4)$	$(1,3)(2,4)$	$(1,4,3,2)$	$(1,4)(2,3)$	$(2,4)$	$(1,2)(3,4)$	$(1,3)$
$(1,2,3,4)$	$(1,2,3,4)$	$(1,3)(2,4)$	$(1,4,3,2)$	e	$(2,4)$	$(1,2)(3,4)$	$(1,3)$	$(1,4)(2,3)$
$(1,3)(2,4)$	$(1,3)(2,4)$	$(1,4,3,2)$	e	$(1,2,3,4)$	$(1,2)(3,4)$	$(1,3)$	$(1,4)(2,3)$	$(2,4)$
$(1,4,3,2)$	$(1,4,3,2)$	e	$(1,2,3,4)$	$(1,3)(2,4)$	$(1,3)$	$(1,4)(2,3)$	$(2,4)$	$(1,2)(3,4)$
$(1,4)(2,3)$	$(1,4)(2,3)$	$(1,3)$	$(1,2)(3,4)$	$(2,4)$	e	$(1,4,3,2)$	$(1,3)(2,4)$	$(1,2,3,4)$
$(2,4)$	$(2,4)$	$(1,4)(2,3)$	$(1,3)$	$(1,2)(3,4)$	$(1,2,3,4)$	e	$(1,4,3,2)$	$(1,3)(2,4)$
$(1,2)(3,4)$	$(1,2)(3,4)$	$(2,4)$	$(1,4)(2,3)$	$(1,3)$	$(1,3)(2,4)$	$(1,2,3,4)$	e	$(1,4,3,2)$
$(1,3)$	$(1,3)$	$(1,2)(3,4)$	$(2,4)$	$(1,4)(2,3)$	$(1,4,3,2)$	$(1,3)(2,4)$	$(1,2,3,4)$	e

.

]

3. Let H be the subgroup of S_5 generated by the elements $r = (1,2,3,4,5)$ and $f = (1,5)(2,4)$.

   (a) Show that $r^5 = e$. [Answer: The order of an $n-$cycle is n so $r^4 = (1,2,3,4,5)^5 = e$.]

   (b) Show that $fr = r^4 f$. [Answer: We know $fr = (1,5)(2,4)(1,2,3,4,5) = (1,4)(2,3)$. From the last question, we know that r^4 must be r^{-1}. Thus $r^4 f = (1,2,3,4,5)^4(1,5)(2,4) = (1,2,3,4,5)^{-1}(1,5)(2,4) = (5,4,3,2,1)(1,5)(2,4) = (1,4)(2,3)$ as well.]

   (c) Show that $fr^k = r^{5-k} f$ for $k \in \{0,1,2,3,4\}$. [Answer: The $k = 0$ case reduces to $f = f$ and the $k = 1$ case has been verified. For the next three cases we could multiply cycles, but we could also use the $k = 1$ case and fact that $r^5 = e$ to see $fr^2 = r^4 fr = r^8 f = r^3 f$, $fr^3 = r^4 fr^2 = r^8 fr = r^{12} f = r^2 f$, and $fr^4 = r^4 fr^3 = r^8 fr^2 = r^{12} fr = r^{16} f = rf$.]

   (d) Write the elements $e, r, r^2, r^3, r^4, f, rf, r^2 f, r^3 f, r^4 f$ in cycle notation. [Answer:

Elt	e	r	r^2	r^3	r^4
Cycle	e	$(1,2,3,4,5)$	$(1,3,5,2,4)$	$(1,4,2,5,3)$	$(1,5,4,3,2)$

Elt	f	rf	$r^2 f$	$r^3 f$	$r^4 f$
Cycle	$(1,5)(2,4)$	$(2,5)(3,4)$	$(1,2)(3,5)$	$(1,3)(4,5)$	$(1,4)(2,3)$

.]

   (e) Make a Cayley table using the set $\{e, r, r^2, r^3, r^4, f, rf, r^2 f, r^3 f, r^4 f\}$. Write each element inside the table in cycle notation.

   [Answer: We split our table into its left and right parts in order to maintain a decent font size.

$\times$	e	r	r^2	r^3	r^4
e	e	$(1,2,3,4,5)$	$(1,3,5,2,4)$	$(1,4,2,3,5)$	$(1,5,4,3,2)$
r	$(1,2,3,4,5)$	$(1,3,5,2,4)$	$(1,4,2,3,5)$	$(1,5,4,3,2)$	e
r^2	$(1,3,5,2,4)$	$(1,4,2,3,5)$	$(1,5,4,3,2)$	e	$(1,2,3,4,5)$
r^3	$(1,4,2,3,5)$	$(1,5,4,3,2)$	e	$(1,2,3,4,5)$	$(1,3,5,2,4)$
r^4	$(1,5,4,3,2)$	e	$(1,2,3,4,5)$	$(1,3,5,2,4)$	$(1,4,2,3,5)$
f	$(1,5)(2,4)$	$(1,4)(2,3)$	$(1,3)(4,5)$	$(1,2)(3,5)$	$(2,5)(3,4)$
rf	$(2,5)(3,4)$	$(1,5)(2,4)$	$(1,4)(2,3)$	$(1,3)(4,5)$	$(1,2)(3,5)$
$r^2 f$	$(1,2)(3,5)$	$(2,5)(3,4)$	$(1,5)(2,4)$	$(1,4)(2,3)$	$(1,3)(4,5)$
$r^3 f$	$(1,3)(4,5)$	$(1,2)(3,5)$	$(2,5)(3,4)$	$(1,5)(2,4)$	$(1,4)(2,3)$
$r^4 f$	$(1,4)(2,3)$	$(1,3)(4,5)$	$(1,2)(3,5)$	$(2,5)(3,4)$	$(1,5)(2,4)$

$\times$	ef	rf	r^2f	r^3f	r^4f
e	$(1,5)(2,4)$	$(2,5)(3,4)$	$(1,2)(3,5)$	$(1,3)(4,5)$	$(1,4)(2,3)$
r	$(2,5)(3,4)$	$(1,2)(3,5)$	$(1,3)(4,5)$	$(1,4)(2,3)$	$(1,5)(2,4)$
r^2	$(1,2)(3,5)$	$(1,3)(4,5)$	$(1,4)(2,3)$	$(1,5)(2,4)$	$(2,5)(3,4)$
r^3	$(1,3)(4,5)$	$(1,4)(2,3)$	$(1,5)(2,4)$	$(2,5)(3,4)$	$(1,2)(3,5)$
r^4	$(1,4)(2,3)$	$(1,5)(2,4)$	$(2,5)(3,4)$	$(1,2)(3,5)$	$(1,3)(4,5)$
f	e	$(1,5,4,3,2)$	$(1,4,2,3,5)$	$(1,3,5,2,4)$	$(1,2,3,4,5)$
rf	$(1,2,3,4,5)$	e	$(1,5,4,3,2)$	$(1,4,2,3,5)$	$(1,3,5,2,4)$
r^2f	$(1,3,5,2,4)$	$(1,2,3,4,5)$	e	$(1,5,4,3,2)$	$(1,4,2,3,5)$
r^3f	$(1,4,2,3,5)$	$(1,3,5,2,4)$	$(1,2,3,4,5)$	e	$(1,3,5,2,4)$
r^4f	$(1,5,4,3,2)$	$(1,4,2,3,5)$	$(1,3,5,2,4)$	$(1,2,3,4,5)$	e

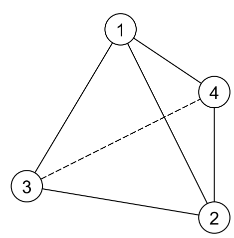
.]

1.3 Groups of Rotational Symmetries

The following questions involve platonic solids and symmetric groups.

1.3.1 The Tetrahedron

Figure 1.26: A Picture of a Tehrahedron



Consider a tehrahedron with corners labeled as in figure 1.26. We wish to examine the group of rotations in three dimensions which map the figure onto itself These rotations may permute faces, corners and edges, but not the directions as a whole that these face.

Any rotation must permute the four labeled corners and two rotations are equivalent if they map corners to the same locations. Because of this we can look at each rotation as an element of S_4. Let T_n be the rotation we get from holding corner n in place and rotating 120 degrees.

1. What element of S_4 do we get from T_1?
   [Answer: $(2,3,4)$.]

2. What element of S_4 do we get from T_1^{-1}
   [Answer: $(2,4,3)$.]

3. Represent each T_i as an element of S_4.
   [Answer: $T_1 = (2,3,4), T_2 = (1,4,3), T_3 = (1,2,4), T_4 = (1,3,2)$.]

4. What are the elements of the group of all rotations by multiples of 120 degrees where corner one is held in place? Note that this is the same as asking which group is generated by T_1, or simply asking you to find $< T_1 >$.
   [Answer: $(\{e, (2,3,4), (2,4,3)\}$.]

5. What is this group isomorphic to?
   [Answer: $\mathbb{Z}_3$.]

6. What group does T_2 generate?
   [Answer: $\{e, (1,3,4), (1,4,3)\}$.]

7. What is the set of all rotations we can get by picking any corner and rotating by any multiple of 120 degrees.
   [Answer: $\{e, (1,2,3), (1,3,2), (1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), (2,4,3)\}$.]

8. Does this contain all possible 3-cycles in S_4?
   [Answer: Yes.]

9. Does this contain all even elements in S_4?
   [Answer: No.]

10. Is this set a group?
    [Answer: No. When taking three cycles from different $< T_i >$ we often end up with two disjoint two cycles. Thus the output will no be from this set. For example $(1,2,3)(2,3,4) = (1,2)(3,4)$.]

11. Is it possible to take two three cycles from different $< T_i >$ and end up with a three cycle?
    [Answer: Yes. For instance, $(1,2,3)(2,4,3) = (1,2,4)$. However, if we replace either one with its inverse, we are back to two two cycles as $(1,3,2)(2,4,3) = (1,3)(2,4)$ and $(1,2,3)(2,3,4) = (1,2)(3,4)$.]

12. Make a table describing the product of any two clockwise rotations.
    [Answer:

$\times$	T_1	T_2	T_3	T_4
T_1	$(2,4,3)$	$(1,2,3)$	$(1,3,4)$	$(1,4,2)$
T_2	$(1,4,2)$	$(1,3,4)$	$(1,2,3)$	$(2,4,3)$
T_3	$(1,2,3)$	$(2,4,3)$	$(1,4,2)$	$(1,3,4)$
T_4	$(1,3,4)$	$(1,4,2)$	$(2,4,3)$	$(1,2,3)$

Recall that $T_1 = (2,3,4), T_2 = (1,4,3), T_3 = (1,2,4), T_4 = (1,3,2)$. This allows us to form the table above.

$\times$	T_1	T_2	T_3	T_4
T_1	T_1^{-1}	T_4^{-1}	T_2^{-1}	T_3^{-1}
T_2	T_3^{-1}	T_2^{-1}	T_4^{-1}	T_1^{-1}
T_3	T_4^{-1}	T_1^{-1}	T_3^{-1}	T_2^{-1}
T_4	T_2^{-1}	T_3^{-1}	T_1^{-1}	T_4^{-1}

We can use that each of these is of the form T_i^{-1} to rewrite the table in a second form as well. Either of these choices is fine.]

13. If we perform two 120 degree clockwise rotations about the corners of two points of the tetrahedron, what can we say about the outcome? Is it always a clockwise rotation, always a counterclockwise rotation, always neither, or does it depend on the two clockwise rotations that we choose?
[Answer: We can use the table we've created to see each of the outputs is an inverse of one of our T_i. Thus the product of any two clockwise rotations is a counterclockwise rotation, no matter which we pick.]

14. Is the set of 120 degree clockwise rotations under composition a group?
[Answer: No. It isn't even a binary operation as it isn't closed. It's not even closed under taking products of the same element, as taking the same clockwise rotation twice still produces a counterclockwise motion. As the table in the last question shows the best way to describe may be as "anti-"closed [2]. For any two elements in the set, their product is never in the set. Thus there are many reasons this is not a binary operation]

15. Make a table describing the product of any two counterclockwise rotations.
[Answer:

$\times$	T_1^{-1}	T_2^{-1}	T_3^{-1}	T_4^{-1}
T_1^{-1}	$(2,3,4)$	$(1,2,4)$	$(1,3,2)$	$(1,4,3)$
T_2^{-1}	$(1,3,2)$	$(1,4,3)$	$(2,3,4)$	$(1,2,4)$
T_3^{-1}	$(1,4,3)$	$(1,3,2)$	$(1,2,4)$	$(2,3,4)$
T_4^{-1}	$(1,2,4)$	$(2,3,4)$	$(1,4,3)$	$(1,3,2)$

Noting that $T_1^{-1} = (2,4,3), T_2^{-1} = (1,3,4), T_3^{-1} = (1,4,2), T_4^{-1} = (1,2,3)$ we get the table above.

$\times$	T_1^{-1}	T_2^{-1}	T_3^{-1}	T_4^{-1}
T_1^{-1}	T_1	T_3	T_4	T_2
T_2^{-1}	T_4	T_2	T_1	T_3
T_3^{-1}	T_2	T_4	T_3	T_1
T_4^{-1}	T_3	T_1	T_2	T_4

As the inner entries are all from the set $\{T_1, T_2, T_3, T_4\}$ we can also write this in a second form.]

16. Is this a group?
[Answer: Again, this is not even a binary operation. None of the outputs are counterclockwise rotations.]

17. If we perform two 120 degree counterclockwise rotations about the corners of two points of the tetrahedron, what can we say about the outcome? Is it always a clockwise rotation, always a counterclockwise rotation, always neither, or does it depend on the two counterclockwise rotations that we choose?
[Answer: We can use the table we've created to see each of the outputs is one of our T_i and hence a clockwise rotation. It doesn't matter which two we pick so the product of any two counterclockwise rotations is always a clockwise rotation.]

[2] Not a real term, but it does fit here.

18. Combining the results of the tables above, make a table describing the results we have found so far for the product of clockwise and counterclockwise rotations. We can do this because the outcome is the same no matter which two we pick. Specifically, we are asking what we get if we let $C = \{T_1, T_2, T_3, T_4\}, CC = \{T_1^{-1}, T_2^{-1}, T_3^{-1}, T_4^{-1}\}$ and make a two by two table describing all we know so far.
[Answer:

$\times$	C	CC
C	CC	
CC		C

We get this table.]

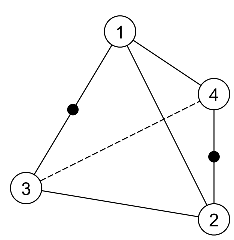
19. If we conjugate a clockwise 120 degree rotation about any point by another clockwise 120 degree rotation, what can we say about the output?
[Answer: Using C to mean clockwise and CC for counterclockwise, we end up with a product of the form $(C)(C)(CC)$. From the above questions, we can't be sure what $(C)(CC)$ may be, but we can resolve the first two elements listed first. We get something of the form $(CC)(CC)$ which we know always gives us a clockwise rotation.]

20. If we conjugate a counterclockwise 120 degree rotation about any point by another counterclockwise 120 degree rotation, what can we say about the output?
[Answer: We always get a counterclockwise rotation.]

21. If we conjugate a clockwise 120 degree rotation about any point by a counterclockwise 120 degree rotation, what can we say about the output?
[Answer: We always get a clockwise rotation.]

22. If we conjugate a counterclockwise 120 degree rotation about any point by a clockwise 120 degree rotation, what can we say about the output?
[Answer: We always get a counterclockwise rotation.]

Figure 1.27: Midpoints of Opposite Sides of a Tehrahedron



23. Consider the midpoints of the 1-3 edge and 2-4 edge as shown in figure . If we form an axis though these midpoints and rotate the tetrahedron 180 degrees, we get a rotation that preserves the position of our object. Consider how this rotation permutes the corners and express this as an element of S_4.
[Answer: $(1, 3)(2, 4)$.]

24. Which permutation do we get by rotating about the midpoints of the 1-2 and 3-4 edges?
[Answer: $(1,2)(3,4)$.]

25. Which permutation do we get by rotating about the midpoints of the 1-4 and 2-3 edges?
[Answer: $(1,4)(2,3)$.]

26. We refer to the last three rotations as flips as they are rotations by a full 180 degrees. Let $F_{a,b}$ be the flip through the midpoint of edge $a-b$ and the other two points. How many distinct flips do we have and what are they?
[Answer: We get only the three we have computed, since there are only six edges and each flip is determined by two. These are $F_{1,2}, F_{1,3}$, and $F_{1,4}$ which are equal to $F_{3,4}, F_{2,4}$, and $F_{2,3}$ respectively.]

27. Find the inverses of $F_{1,2}, F_{1,3}$, and $F_{1,4}$.
[Answer: As each is a 180 degree rotation, performing any twice returns the tetrahedron to its original position. Thus each of these are their own inverse. We can also see this by multiplying the elements in cycle notation and noting that disjoint cycles commute. Thus $F_{1,2}^2 = (1,3)(2,4)(1,3)(2,4) = (1,3)(1,3)(2,4)(2,4) = e \cdot e = e$ and $F_{1,3}^2$ and $F_{1,4}^2$ are e as well.]

28. Find $F_{1,2} \cdot F_{1,3}$.
[Answer: $(1,2)(3,4)(1,3)(2,4) = (1,4)(2,3) = F_{1,4}$.]

29. Make a multiplication table describing the composition of the elements in the set $F = \{e, F_{1,2}, F_{1,3}, F_{1,4}\}$.

$\times$	e	$F_{1,2}$	$F_{1,3}$	$F_{1,4}$
e	e	$F_{1,2}$	$F_{1,3}$	$F_{1,4}$
$F_{1,2}$	$F_{1,2}$	e	$F_{1,4}$	$F_{1,3}$
$F_{1,3}$	$F_{1,3}$	$F_{1,4}$	e	$F_{1,2}$
$F_{1,4}$	$F_{1,4}$	$F_{1,3}$	$F_{1,2}$	e

30. Is this table the Cayley table of some group? If so, which group is it isomorphic to?
[Answer: Yes. This is isomorphic to the Klein four group V (which is also isomorphic to $U(8)$, D_2, $\mathbb{Z}_2^2$ and many other groups we have studied.]

31. Make a table showing what we get when we multiply a flip by a clockwise rotation.
[Answer:

$\times$	T_1	T_2	T_3	T_4
e	T_1	T_2	T_3	T_4
$F_{1,2}$	T_3	T_4	T_1	T_2
$F_{1,3}$	T_4	T_1	T_2	T_3
$F_{1,4}$	T_1	T_3	T_2	T_1

We can do this by actually twisting an actual tetrahedron, or by multiplying out terms in cycle notation. For example: $F_{1,2}T_1 = (1,2)(3,4)(2,3,4) = (1,2,4) = T_3$,$F_{1,2}T_2 = (1,2)(3,4)(1,4,3) = (1,3,2) = T_4$, $F_{1,2}T_3 = (1,2)(3,4)(1,2,4) = (2,3,4) = T_1$, and so on. We get the table above.]

32. Make a table showing what we get when we multiply a flip by a counterclockwise rotation.
[Answer:

$\times$	T_1^{-1}	T_2^{-1}	T_3^{-1}	T_4^{-1}
e	T_1^{-1}	T_2^{-1}	T_3^{-1}	T_4^{-1}
$F_{1,2}$	T_4^{-1}	T_3^{-1}	T_2^{-1}	T_1^{-1}
$F_{1,3}$	T_2^{-1}	T_1^{-1}	T_4^{-1}	T_3^{-1}
$F_{1,4}$	T_3^{-1}	T_4^{-1}	T_1^{-1}	T_2^{-1}

Again we can do this by actually twisting an actual tetrahedron, or by multiplying out terms in cycle notation. For example: $F_{1,2}T_1^{-1} = (1,2)(3,4)(2,4,3) = (1,2,3) = T_4^{-1}$,$F_{1,2}T_2 = (1,2)(3,4)(1,3,4) = (1,4,2) = T_3^{-1}$, $F_{1,2}T_3 = (1,2)(3,4)(1,4,2) = (1,3,4) = T_2^{-1}$ We get the table above.]

33. What sort of rotation do we get when we multiply an element of F by a clockwise rotation?
[Answer: We always get another clockwise rotation.]

34. What sort of rotation do we get when we multiply an element of F by a counterclockwise rotation?
[Answer: We always get another counterclockwise rotation.]

35. What is the parity of each of the elements in F?
[Answer: All are even permutations in S_4.]

36. What is the parity of each of the elements in C and CC?
[Answer: All are even permutations in S_4.]

37. What is the size of $C \cup CC \cup F$?
[Answer: As these are disjoint sets of permutations, and each has size four, the cardinality is twelve.]

38. What is the size of A_4?
[Answer: $S_4/2 = 24/2 = 12$.]

39. Prove that $C \cup CC \cup F = A_4$
[Answer: $C \cup CC \cup F$ is a set of even permutations of size twelve. There are only twelve even permutations in S_4. Thus the two must be the same.]

40. Combining all we know so far, make a table describing the results we have found so far for the product of clockwise rotations, counterclockwise rotations and the motions in the set F. We can do this because the outcome is the same no matter which elemets of those sets we pick.
[Answer:

$\times$	F	C	CC
F	F	C	CC
C	C	CC	F
CC	CC	F	C

We get the table above.]

41. Is this the Cayley table of some group? If so, what is the identity and what group is this isomorphic to?
[Answer: Yes. F is the identity element and it is isomorphic to $\mathbb{Z}_3$.

42. We have seen that all the 120 degree rotations, 180 degree flips, and the identity together form a group. We have also seen this group is isomorphic to A_4. However, is it possible that there is some rotation we are missing and thus our group is a larger subgroup of S_4?
[Answer: No. The rotations form a subgroup. By Lagrange, the order of any subgroup divides the order of the group. So if there were more rotations, the only possibility for the group of rotations would be all of S_4. That contains all the two cycles. We know it is not possible to interchange just two corners of a tetrahedron without moving the rest, thus this cannot happen.]

43. Is there a subgroup of the group of rotational symmetries of the tetrahedron that has order two? If so, then state how many and describe the rotations in this group instead of writing them out as cycles.
[Answer: Yes. There are three. Pick two opposite edges and reflect through the midpoints of those edges by 180 degrees. This generates a subgroup of order two.]

44. Is there a subgroup of the group of rotational symmetries of the tetrahedron that has order three? If so, then state how many and describe the rotations in this group instead of writing them out as cycles.
[Answer: Yes. There are four. Pick any corner and rotate by 120 degrees. This generates a subgroup of order three.]

45. Is there a subgroup of the group of rotational symmetries of the tetrahedron that has order four? If so, then state how many and describe the rotations in this group instead of writing them out as cycles.
[Answer: Yes. There is only one. Note that no subgroup of order four can have any of elements of order three. Thus all the 120 degree rotations are out. This leaves only the four elements that come from the identity or rotating 180 degrees through the midpoints of opposite edges. As this is isomorphic to the Klein-four group, this is a subgroup of order four. As these are the only four elements that can be in such a subgroup, this is our only option.]

46. Is there a subgroup of the group of rotational symmetries of the tetrahedron that has order five? If so, then state how many and describe the rotations in this group instead of writing them out as cycles.
[Answer: No. Be Lagrange, the order of the subgroup needs to divide the order of the group.]

47. Is there a subgroup of the group of rotational symmetries of the tetrahedron that has order six? If so, then state how many and describe the rotations in this group instead of writing them out as cycles.
[Answer: No. As six divides twelve, there might be a subgroup of order six according to Lagrange, but the theorem does not guarantee us a subgroup of order six. A subgroup of order six would contain elements of order 2 and elements of order three, as both D_3 and $\mathbb{Z}_6$ have these. Thus we'd need to include rotations of degree two and degree three as in any subgroup of order six. If we conjugate a 120 degree rotation by a flip we get a 120 degree rotation about a different point. Thus for closure we'd need at least four elements of order three, which no group of order six has.]

1.3.2 The Cube

Consider a cube with corners labeled as in figure 1.28. We wish to examine the group of rotations in three dimensions which map the figure onto itself These rotations may permute faces, corners and edges, but not the directions as a whole that these face.

Figure 1.28: A Picture of a Cube

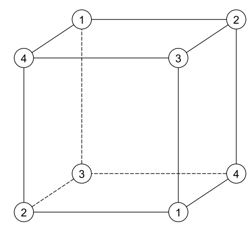
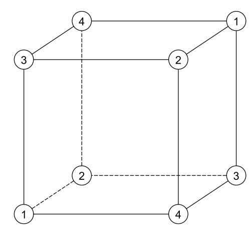


Figure 1.29: A Picture of a Cube



For example, if we rotate the top of the cube ninety degrees clockwise we get the following image shown in figure 1.29. Notice that both corners labeled one go to corners labeled two, and both corners labeled two go to ones labeled three, and so on. Because of that we can think of this rotation as a permutation of the set $\{1, 2, 3, 4\}$ and an element of S_n. [3]

1. What element of S_4 do we get from the rotation described in figure 1.29.
   [Answer: $(1, 2, 3, 4)$.]

2. What element of S_4 do we get from rotating the top of cube 180 degrees clockwise?
   [Answer: $(1, 3)(2, 4)$.]

3. What element of S_4 do we get from rotating the top of cube 90 degrees counterclockwise (and hence the bottom 90 degrees clockwise?)
   [Answer: $(1, 4, 3, 2)$.]

4. Consider the group of rotations of the top and bottom of the cube by multiples of ninety degrees. What elements of S_4 do we get? [$\{e, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2)$.]

5. What is this subgroup isomorphic to? [$\mathbb{Z}_4$.]

6. Consider the rotation of the front of the cube by ninety degrees clockwise (and hence the back of the cube ninety degrees counterclockwise. What element of S_4 do we get from this rotation?
   [Answer: $(1, 2, 4, 3)$.]

[3]If at any time we encountered a rotation that didn't map both corners with one number to corners with the same number, none of this would work. Don't worry. That can't happen.

7. Consider the subgroup generated by this element. This is the subgroup of rotations of the front and back sides by multiples of ninety degrees. Which elements of S_4 does this contain?
   [Answer: $\{e, (1, 2, 4, 3), (1, 4)(2, 3), (1, 3, 4, 2)\}$.]

8. Consider the rotation of the right of the cube by ninety degrees clockwise (and hence the left of the cube ninety degrees counterclockwise. What element of S_4 do we get from this rotation?
   [Answer: $(1, 3, 2, 4)$.]

9. Consider the subgroup generated by this element. This is the subgroup of rotations of the right and left sides by multiples of ninety degrees. Which elements of S_4 does this contain?
   [Answer: $\{e, (1, 3, 2, 4), (1, 2)(3, 4), (1, 4, 2, 3)\}$.]

10. What is the set of all rotations, clockwise or counterclockwise, of any side by ninety degrees?
    [Answer: $\{(1, 2, 3, 4), (1, 4, 3, 2), (1, 2, 4, 3), (1, 3, 4, 2), (1, 3, 2, 4), (1, 4, 2, 3)\}$.]

11. Is this set a group?
    [Answer: It's not even a binary operation. It's not closed under the product as two ninety degree rotations make a 180 degree rotation, and none of those are contained in the set.]

12. What is the set of all rotations, clockwise or counterclockwise, of any side by any multiple of ninety degrees?
    [Answer: $\{e, (1, 2, 3, 4), (1, 4, 3, 2), (1, 3)(2, 4), (1, 2, 4, 3), (1, 3, 4, 2), (1, 4)(2, 3), (1, 3, 2, 4), (1, 4, 2, 3), (1, 2)(3, 4)\}$.]

13. Is this set a group? If so, state what it is isomorphic to.
    [Answer: No. It is not closed as $(1, 2, 3, 4)(1, 2)(3, 4) = (1, 3)$.]

14. What is the set of all rotations, clockwise or counterclockwise, of any side by any multiple of 180 degrees?
    [Answer: $\{e, (1, 3)(2, 4), (1, 4)(2, 3), (1, 2)(3, 4)\}$.]

15. Is this set a group? If so, state what it is isomorphic to.
    [Answer: Yes. Each of the nontrivial elements has order two so this must be the Klein-four group.]

16. What do we get if we conjugate a 90 degree clockwise top side rotation by a 90 degree clockwise right side rotation?
    [Answer: A 90 degree counterclockwise front side rotation. We can do this with cycle notation to get $(1, 3, 2, 4)(1, 2, 3, 4)(1, 4, 2, 3) = (1, 3, 4, 2)$. This is also easily done by manipulating a physical cube if we remember that we act from right to left on the object.]

17. Consider the axis about both of the corners labeled one. Take the collection of all rotations about this axis by any multiple of 120 degrees. Is this a group? If so, list the elements and state what it is isomorphic to.
    [Answer: It is the group $\{e, (2, 3, 4), (2, 4, 3)\}$. This is isomorphic to $\mathbb{Z}_3$.]

18. Consider the axis about both of the corners labeled two. Take the collection of all rotations about this axis by any multiple of 120 degrees. Is this a group? If so, list the elements and state what it is isomorphic to.
    [Answer: It is the group $\{e, (1, 3, 4), (1, 4, 3)\}$. This is isomorphic to $\mathbb{Z}_3$.]

19. Consider the axis about both of the corners labeled three. Take the collection of all rotations about this axis by any multiple of 120 degrees. Is this a group? If so, list the elements and state what it is isomorphic to.
[Answer: It is the group $\{e, (1,4,2), (1,2,4)\}$. This is isomorphic to $\mathbb{Z}_3$.]

20. Consider the axis about both of the corners labeled four. Take the collection of all rotations about this axis by any multiple of 120 degrees. Is this a group? If so, list the elements and state what it is isomorphic to.
[Answer: It is the group $\{e, (1,3,2), (1,2,3)\}$. This is isomorphic to $\mathbb{Z}_3$.]

21. What is the set of all rotations about any opposite corners by any multiple of 120 degrees?
[Answer: $\{e, (2,3,4), (2,4,3), (1,3,4), (1,4,3), (1,4,2), (1,2,4), (1,3,2), (1,2,3)\}$]

22. Is this set a group? If so, state what it is isomorphic to.
[Answer: No. It is not closed. For example $(1,2,3)(2,3,4) = (1,2)(3,4)$ which is not a 120 degree rotation.]

23. What is the set consisting of the identity, all rotations about any opposite corners by any multiple of 120 degrees, and all 180 degree rotations about any sides?
[Answer: $\{e, (2,3,4), (2,4,3), (1,3,4), (1,4,3), (1,4,2), (1,2,4), (1,3,2), (1,2,3), (1,3)(2,4), (1,4)(2,3), (1,2)(3,4)$

24. Is this set a group? If so, state what it is isomorphic to.
[Answer: Yes. Each of the elements is even, and we have twelve distinct elements. S_4 only has twelve even elements and thus we have found all of them. Thus we have obtained all the elements of the group A_4.]

25. Consider the axis about the midpoints of both edges going from corners labeled one and two. What element do we get if we rotate about this axis by ninety degrees?
[Answer: $(3,4)$.]

26. Consider the axis about the midpoints of both edges going from corners labeled three and four. What element do we get if we rotate about this axis by ninety degrees?
[Answer: $(1,2)$.]

27. Can we achieve any 2-cycle in S_4 by some rotation of this labeled cube?
[Answer: Yes. In order to get the two cycle (a,b), just let $\{c,d\}$ be the compliment of $\{a,b\}$ and take the 180 degree rotation about the midpoints of the $c-d$ edge.]

28. Prove that the group of rotational symmetries contains S_4.
[Answer: We have shown it contains all the two cycles, and the two cycles generate all of S_4.]

29. Prove that the S_4 permutations contain all the rotational symmetries of the cube.
[Answer: Picture the four axes through opposite corners of the cube. Any rotation bringing the cube to the same space must map axes to axes. Thus any rotational symmetry results in some permutation of S_4. For a second, completely different argument, note that any rotation maps a side of the cube to one of six locations, and once there that side can only be rotated into four possibilities. Thus there are a maximum of 24 elements in the group of all rotations. As we have already shown the group contains 24 messages, the group of rotations must equal S_4 exactly.]

30. Which rotations of the cube amount to odd permutations in S_4?
[Answer: The 180 degree rotations about the midpoints of opposite edges, together with the ninety degree rotations about one face.]

31. Which rotations of the cube amount to even permutations in S_4?
[Answer: The 120 degree rotations about opposite corners, together with the 180 degree rotations about one face and the identity.]

32. Consider the set containing the identity, rotations about the midpoint of the one two edges, the midpoint of the three four edges, and the composition of these two rotations. What elements are contained in this set?
[Answer: $\{e, (3,4), (1,2), (1,2)(3,4)\}$.]

33. Is this set a group? If so, state what it is isomorphic to.
[Answer: Yes. It is isomorphic to the Klein-four group.]

34. Consider the set containing the identity, all rotations of the top of the cube by multiples of ninety degrees, the flip about the midpoints of the two-four edge, and any combination of these. What elements are contained in this set?
[Answer: $\{e, (1,2,3,4), (1,3)(2,4), (1,4,3,2), (1,3), (1,4)(2,3), (2,4), (1,2)(3,4)\}$.]

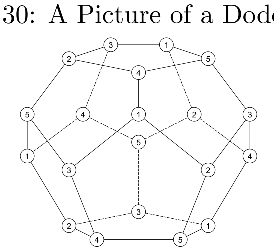
35. Is this set a group? If so, state what it is isomorphic to.
[Answer: Yes. It is isomorphic to D_8.]

36. Consider the set containing the identity, all rotations about the axis through the corners labeled four by multiples of 120 degrees, all 180 degree rotations through the midpoint of the three-four edges, and every combination of these.What elements are contained in this set?
[Answer: $\{e, (1,2,3)(1,3,2), (1,2), (1,3), (2,3)\}$.]

37. Is this set a group? If so, state what it is isomorphic to.
[Answer: Yes. It is isomorphic to S_3 or D_3.]

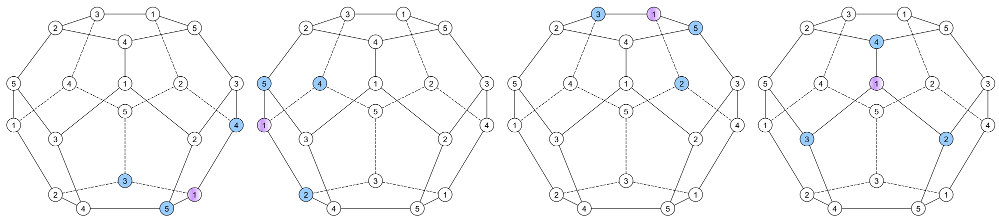
1.3.3 The Dodecahedron

Figure 1.30: A Picture of a Dodecahedron



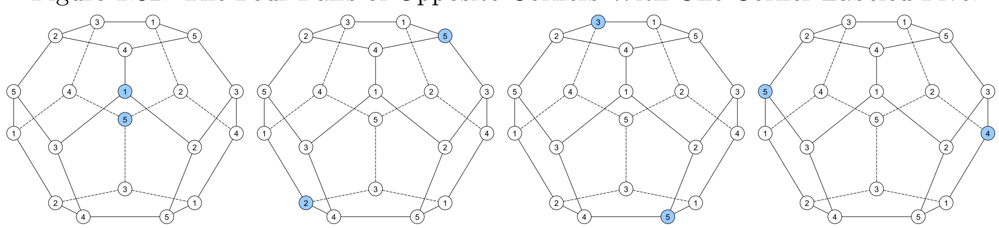
Consider a dodecahedron with corners labeled as in figure 1.30. We wish to examine the group of rotations in three dimensions which map the figure onto itself These rotations may permute faces, corners and edges, but not the directions as a whole that these face. For most of these questions, it may really help to make a physical dodecahedron and to draw the numbers onto the same corners.

Figure 1.31: The Four Cornered Labeled One.



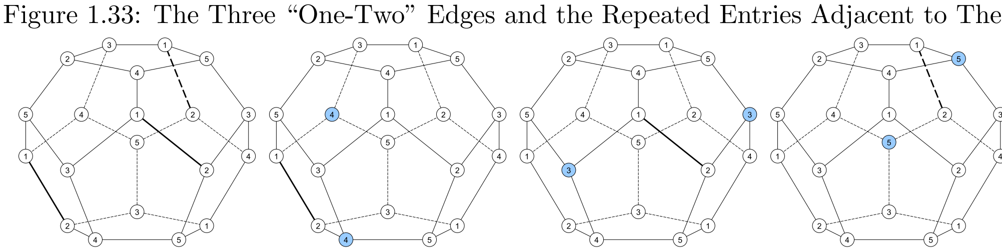
Consider any given corner in a dodecahedron with corners labeled as in figure 1.30. The three distinct corners adjacent to it are labeled with three numbers distinct from each other and the original corner. Each label has four possibilities for three distinct numbers not equal to it, and indeed each of these possibilities appears exactly once. Thus we can describe any corner by stating its label and the labels of the three corners adjacent to it. In figure 1.31 we show the one labeled corners with adjacent entries from $\{3,4,5\}, \{2,4,5\}, \{2,3,5\}$, and $\{2,3,4\}$ respectively.

Figure 1.32: The Four Pairs of Opposite Corners With One Corner Labeled Five.



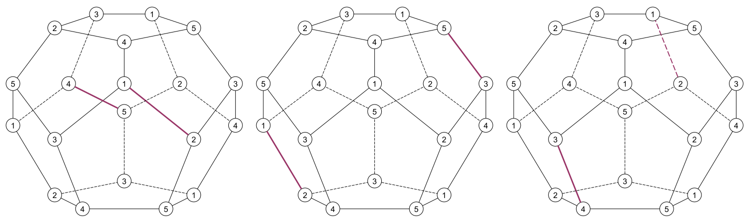
For any corner, the opposite corner is also distinct from the both original corner and all of the original corner's neighbors. There are ten pairs of opposite corners, which is equal to five choose two. Every possibility of two distinct numbers appears once as the labels for some set of opposite corners. Thus we can describe any pair of opposite corners with a pair of distinct numbers. Figure 1.32 shows the pairs of opposite corners involving five and each of the four possibilities.

Figure 1.33: The Three "One-Two" Edges and the Repeated Entries Adjacent to Them.



Any edge is labeled with two distinct numbers and any two distinct labels belong to exactly three of the edges. For each of the corners on these edges, the four corners adjacent to them have three distinct entries with one repeated. Every possibility of the remaining three appears exactly once as the repeated entry. For example if we look at the three one-two edges, there is one with repeated threes, one with repeated fours and one with repeated fives. We can see these three edges in figure 1.33.

Figure 1.34: The Three "One-Two" Edges and the Opposite Edges with Each Possible Values.



Finally, for any combination of two distinct numbers there are three edges with those numbers as endpoints. Each of those three edges have an opposite edge, each of those edges contain numbers distinct from the original two, and every possibility of distinct pairs appears. Because of this, we can describe each of the fifteen sets of opposite corners by designating two numbers for one edge and a different two of the remaining numbers for the opposite edge. We can see an example of this in figure 1.34.

Figure 1.35: A Picture of a Dodecahedron

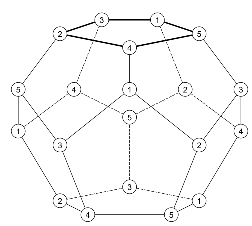
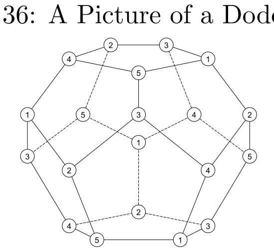


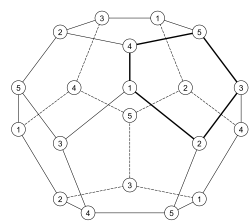
Figure 1.36: A Picture of a Dodecahedron



Consider a rotation clockwise 72 degrees about the face shown in figure 1.35. Notice that each of the corners labeled one get mapped to a corner labeled five. Each of the corners labeled five go to a corner labeled four. For any number, all of the corners labeled with one number go to corners all labeled another number. The image we get after rotation is shown in figure 1.36. Because of this, we can think of this and the other rotations we are considering, as permutations of a set of five elements. Of course, if we ever encountered a rotation that didn't map all of one number to all of one number, then this series of arguments would not work, but fortunately for us this does not happen.

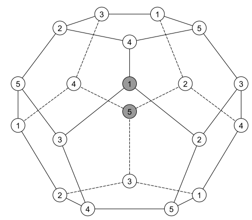
1. Which element of S_5 do we get from the rotation shown in figure 1.36 generated by rotating 72 degrees clockwise about the face in figure 1.35?
   [Answer: $(1, 5, 4, 2, 3)$.]
2. Which element of S_5 do we get by rotating 72 degrees counterclockwise about the face in figure 1.35?
   [Answer: $(1, 3, 2, 4, 5)$.]
3. What is the set of all permutations we get from rotating about the face in figure 1.35 by any multiple of 72 degrees?
   [Answer: $\{e, (1, 5, 4, 2, 3), (1, 4, 3, 5, 2), (1, 2, 5, 3, 4), (1, 3, 2, 4, 5)\}$.]
4. Does this set form a group? If so what well known group is it isomorphic to?
   [Answer: Yes. It is isomorphic to $\mathbb{Z}_5$.]

Figure 1.37: A Picture of a Dodecahedron



5. Which element of S_5 do we get by rotating 72 degrees clockwise about the face in figure 1.37?
   [Answer: $(1, 4, 5, 3, 2)$.]
6. Which element of S_5 do we get by rotating 72 degrees counterclockwise about the face in figure 1.37?
   [Answer: $(1, 2, 3, 5, 4)$.]
7. What is the set of all permutations we get from rotating about the face in figure 1.37 by any multiple of 72 degrees?
   [Answer: $\{e, (1, 4, 5, 3, 2), (1, 5, 2, 4, 3), (1, 3, 4, 2, 5), (1, 2, 3, 5, 4)\}$.]
8. Does this set form a group? If so what well known group is it isomorphic to?
   [Answer: Yes. Again, this group is isomorphic to $\mathbb{Z}_5$.]
9. Is the product of the two clockwise 72 degree rotations about the faces shown in figures 1.35 and 1.37 in any order a 72 degree rotation about a face in either direction?
   [Answer: $(1, 5, 4, 2, 3)(1, 4, 5, 3, 2) = (1, 2, 5)$ which is an element of order three. Any 72 degree rotation would have to have order five. Similarly, $(1, 4, 5, 3, 2)(1, 5, 4, 2, 3) = (1, 3, 4)$ so this also cannot be a 72 degree rotation. Thus the answer is no.]
10. How many non-trivial elements of the group of rotational symmetries arise from rotations about opposite faces of some multiple of 72 degrees?
    [Answer: There are six pairs of opposite faces, each generating four different five cycles and the identity. Thus there are four times six or twenty-four different elements.]
11. How many of these result in even permutations of S_5?
    [Answer: All of them.]

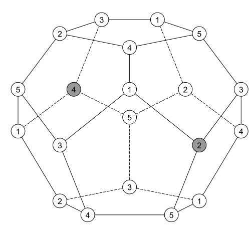
Figure 1.38: A Picture of a Dodecahedron With Two Distinguished Corners



12. Which element of S_5 do we get by rotating 120 degrees clockwise about the axis through the two shaded corners shown in figure 1.38? Note that this is the unique pair of corners labeled with a one and five, so we could have simply described the axis that way as well.
[Answer: $(2, 3, 4)$.]

13. Which element of S_5 do we get by rotating 120 degrees counterclockwise about the axis through the two shaded corners shown in figure 1.38?
[Answer: $(2, 4, 3)$.]

Figure 1.39: A Picture of a Dodecahedron With Two Distinguished Corners



14. Which element of S_5 do we get by rotating 120 degrees clockwise about the axis through the two shaded corners shown in figure 1.39? Note that this is the unique pair of corners labeled with a two and four, so we could have simply described this as a rotation through the two-four axis.
[Answer: $(1, 3, 5)$.]

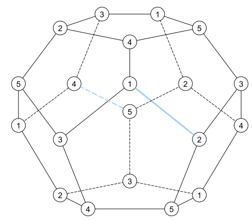
15. Which element of S_5 do we get by rotating 120 degrees counterclockwise about the axis through the two shaded corners shown in figure 1.39?
[Answer: $(1, 5, 3)$.]

16. What permutation do we get when we first rotate clockwise about the axis from figure 1.38 and then rotate about the axis in figure 1.39?
[Answer: $(1, 3, 5)(2, 3, 4) = (1, 3, 4, 2, 5)$.]

17. Does this permutation correspond to a 72 rotation about some face in some direction?
[Answer: No.]

18. Does this permutation correspond to a multiple of some rotation about some face in some direction?
[Answer: Yes. There are two opposite faces with the labels 1-2-3-5-4 in order, one clockwise and one counter clockwise. It corresponds to rotating those faces 144 degrees.]

Figure 1.40: A Picture of Opposite Edges of a Dodecahedron



19. How many elements of the group of rotational symmetries arise from 120 degree rotations in some direction about opposite corners?
[Answer: There are ten pairs of opposite corners, each with two directions we can go in. We get twenty such elements, as each of these possibilities is distinct.]

20. How many of these result in even permutations of S_5.
[Answer: Each is a three cycle, so the answer is all of them.]

21. Consider the 180 degree rotation through the midpoints of the edges shown in figure 1.40. What permutation of S_5 do we get from this rotation.
[Answer: $(1,2)(4,5)$.]

22. How many elements of the group of rotational symmetries arise from 180 degree rotations about the midpoints of opposite edges?
[Answer: Each combination of opposite edges gives a different rotation, and there are fifteen pairs of opposite edges, thus the answer is fifteen.]

23. What is the set of distinct permutations in S_5 arising from 180 degree rotations about the midpoints of opposite edges?
[Answer: $\{(1,2)(3,4)$, $(1,3)(2,4)$, $(1,4)(2,3)$, $(1,2)(3,5)$, $(1,3)(2,5)$, $(1,5)(2,3)$, $(1,2)(4,5)$, $(1,4)(2,5)$, $(1,5)(2,4)$, $(1,4)(3,5)$, $(1,3)(4,5)$, $(1,5)(4,3)$, $(3,5)(4,2)$, $(2,5)(4,3)$, $(2,3)(4,5)\}$.]

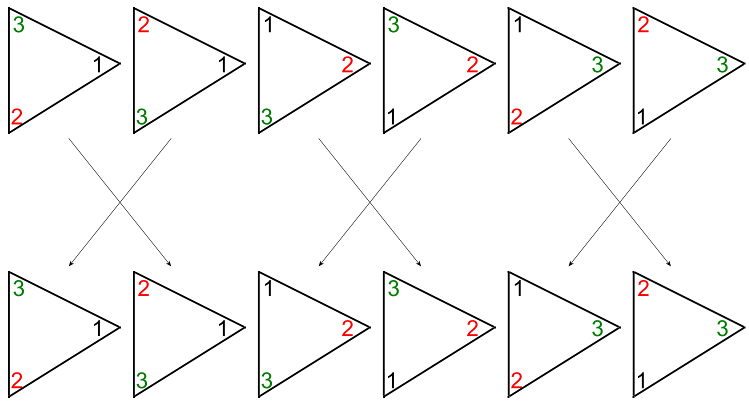
24. Prove that every element of A_5 is contained in the group of rotations of the dodecahedron.
[Answer: The elements we've found so far include one identity, twenty-four face rotations, twenty opposite corner rotations, and fifteen opposite edge rotations. These are sixty permutations all of which are even. There are only sixty even permutations in S_5, thus we must have all of A_5.]

25. Prove that every element of the group of rotations of the dodecahedron is contained in A_5.
[Answer: We must show there are no more than the sixty we've found. Any rotation maps a given face to one of twelve faces, and once that face is chosen, there are only five possibilities for how it can be placed. Thus there cannot be more than sixty permutations.]

1.4 Group Actions

We initially defined the dihedral groups by considering the collection of transformations of some object. We can also take the set of all possible positions of our object, and view every element of the group as a

Figure 1.41: The element f of D_4 as a permutation of the six labeled triangles shown.



permutation of this set. For example, by flipping them over a horizontal line of reflection, the element f in D_3 permutes the six labeled triangles as shown in figure 1.41.

If we have an existing group G and set X, we can try to visualize the elements of the group as permutations of that set, in such a way that makes sense given the structure of the group. It seems reasonable to ask that the identity fix all elements of the set, and that the product of the group is in some sense preserved. If each individual element $g \in G$ is to function as a permutation of X, then for any g and any element x, we can map that x through the permutation for g to get a new x. Thus we need a map that takes in a g and x, and spits out an x. We put this together with the properties we desire and form the following definition.

A group action of the group G on the set X is a map $G \times X \to X$ which satisfies the following:

1. $e \cdot x = x$ for any $x \in X$.

2. $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and $x \in X$.

For a first example consider the set $\{(a,b,c) : a,b,c \in \mathbb{Z}_3, a \neq b, a \neq c, b \neq c\}$ which equals $\{(0,1,2),$ $(0,2,1)$, $(1,0,2)$, $(1,2,0)$, $(2,0,1)$, $(2,1,0)\}$. We can define an action on $\mathbb{Z}_3$ and X by setting $x \cdot (a,b,c) = (a+x, b+x, c+x)$ where the addition takes place in $\mathbb{Z}_3$. Note that as the identity is zero, $e \cdot (a,b,c) = (a+0, b+0, c+0) = (a,b,c)$. Also if we take $g \cdot (h \cdot (a,b,c))$ for any g and h in $\mathbb{Z}_3$, we get $g \cdot (a+h, b+h, c+h) = ((a+h)+g, (b+h)+g, (c+h)+g) = (a+(h+g), b+(h+g), c+(h+g)) = (g \cdot h) \cdot (a,b,c)$. Thus both of our criterion have been met. We

This is not the only way we can define an action of $\mathbb{Z}_3$ on this X. We can define a different action by setting $g \cdot (a_1, a_2, a_3)$ to be $(a_{1-g}, a_{2-g}, a_{3-g})$ where the addition in the indeces takes place modulo three and the least positive reside is chosen. Thus the identity fixes everything, the element 1 rotates the entries one slot to the left, and the element two rotates the entries one slot to the right. This also meets the criteria for being a group action, and the elements of $\mathbb{Z}_3$ form the permutations shown in figure 1.43

For both the idea of adding to entries, and shifting them to the right, the group $\mathbb{Z}_3$ seems like a natural choice. If we tried $\mathbb{Z}_2$ and $x \cdot (a,b,c) = (a+x, b+x, c+x)$, we would not be able to meet the criteria. For one example of this note that $1 \cdot 1 \cdot (a,b,c) = 1 \cdot (a+1, b+1, c+1) = (a+2, b+2, c+2)$ but $(1 \cdot 1) \cdot (a,b,c) = 0 \cdot (a,b,c) = (a,b,c)$. The two are not the same, so this fails to be a group action. If we tried the same thing with shifting the entries, we run into the same problem. We can still find a way to make $\mathbb{Z}_2$ act sensibly, but we need a permutation where if we apply the element one twice, we get the

Figure 1.42: The elements of Z_3 as permutations of our set X.

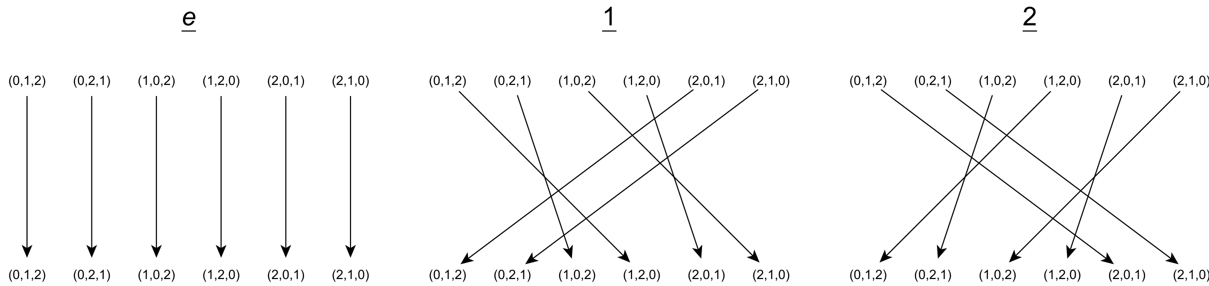
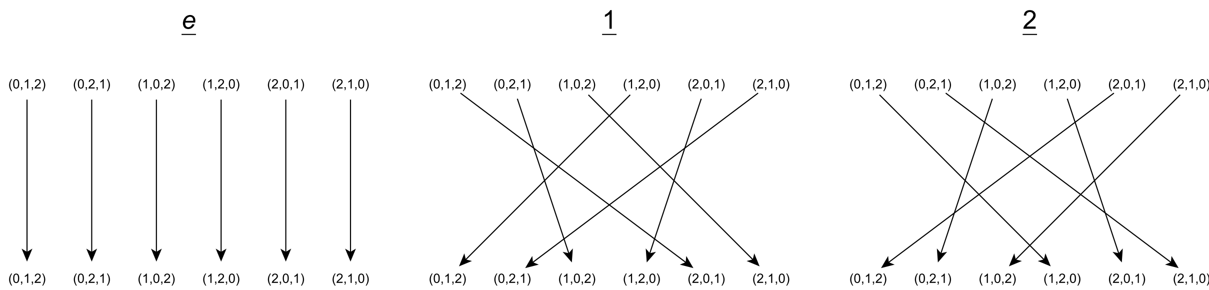
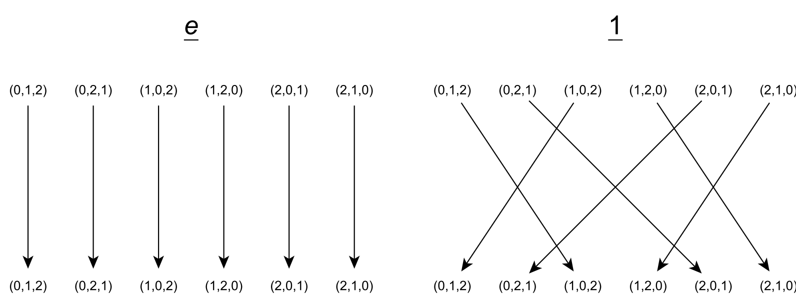


Figure 1.43: The elements of Z_3 as permutations of our set X in a different action.



identity. One way to do this is to set $1 \cdot (a, b, c) = (b, a, c)$ and $0 \cdot (a, b, c) = (a, b, c)$. We can check that this meets the criterion for a group action and it results in the permutations shown in figure

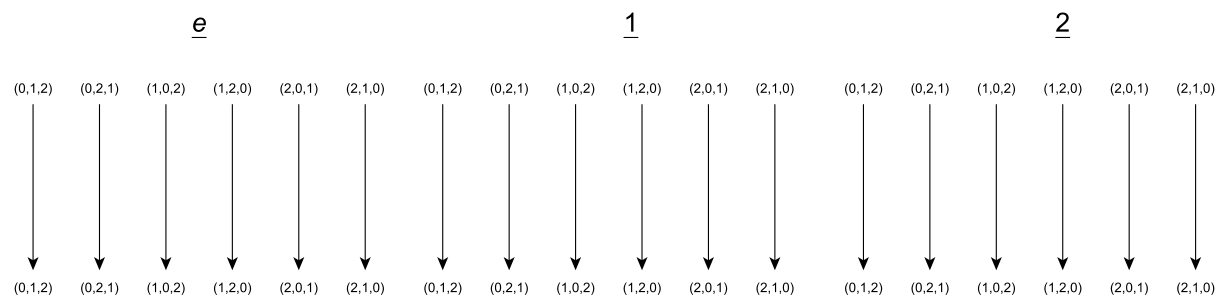
Figure 1.44: A Z_2 action on the same set X.



For a set, we generally want to think about which groups might result in permutations that feel natural. There is one way to define a group action with any group and any set. Simply let $g \cdot x = x$ for every $g \in G$ and $x \in X$. All the axioms are met, and every element of the group becomes the identity permutation on X.

This is called the trivial action, and figure 1.45 illustrates this for our X and $G = \mathbb{Z}_3$.

Figure 1.45: The trivial Z_3 action on the set X.

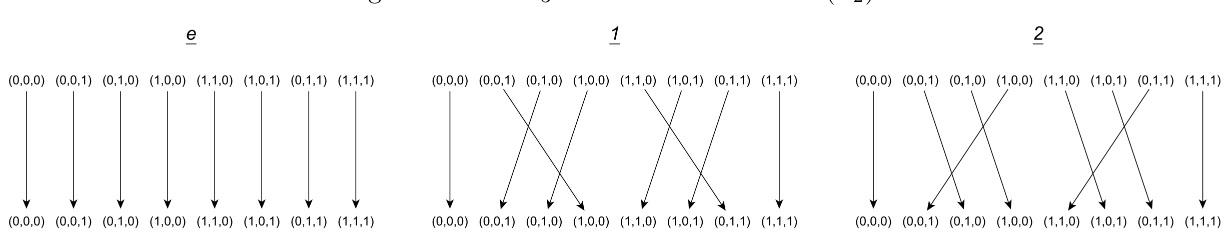


We define the orbit of x in X to be the set $orb(x) = \{gx : g \in G\}$, and the stabilizer to be the set $stab(x) = \{g \in G : gx = x\}$. For any g in G we define the fixed points of g to be the set $fix(g) = X^g = \{x \in X : gx = x\}$. Setting $x \sim y$ if $x \in orb(y)$ produces an equivalence relation, and thus the set of distinct orbits partitions the set X.

For an example, consider the set of all triples in $\mathbb{Z}_2$ which equals $\{(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1).\}$ We construct a $G = \mathbb{Z}_3$ action on this set by allowing G to shift entries to the right by a number of steps equal to the element. Thus $g{\cdot}(a_1, a_2, a_3)$ to be $(a_{1+g}, a_{2+g}, a_{3+g})$. Here the orbit of $(0,0,0) \in X$ is just $\{(0,0,0)\}$ as any amount of shifting leaves this element fixed. Similarly, $orb((1,1,1)) = \{(1,1,1)\}$. We get $orb((0,0,1)) = \{(0,0,1),(0,1,0),(1,0,0)\}$ and $orb((0,1,1)) = \{(0,1,1),(1,1,0),(1,0,1)\}$. As shifting doesn't change the number of ones and zeroes, we get the same orbit from any element containing the same number of zeroes. Thus $orb((0,0,1)) = orb((0,1,0)) = orb((1,0,0))$ and $orb((0,1,1)) = orb((1,0,1)) = orb((1,1,0))$. Any element containing both ones and zeroes will look different after shifting. Thus $stab((0,0,1)) = stab((0,1,0)) = stab((1,0,0)) = stab((0,1,1)) = stab((1,0,1)) = stab((1,1,0)) = \{e\}$. The elements containing all ones or all zeros are unchanged by all of G so $stab((0,0,0)) = stab((1,1,1)) = G$.

In can help to draw out the permutation as shown in figure 1.46. If we look at an individual x then to compute the orbit, we find the collection of all possible images of that x in any of the maps. To compute the stabilizer, we take the collection of maps that leave that element unchanged.

Figure 1.46: A Z_3 action on the set $X = (\mathbb{Z}_2)^3$.



1. The following questions involve the $\mathbb{Z}_3$ group action on the set $\{(a,b,c) : a,b,c \in \mathbb{Z}_3, a \neq b, a \neq c, b \neq c\}$ defined by $x \cdot (a,b,c) = (a+x, b+x, c+x)$.

(a) Find the orbit of each element of X. [Answer: $orb((0,1,2)) = orb((1,2,0)) = orb((2,0,1)) = \{(0,1,2),(1,2,0),(2,0,1)\}$ and $orb((0,2,1)) = orb((1,0,2)) = orb((2,1,0)) = \{(0,2,1),(1,0,2),(2,1,0)\}$.]

(b) Find the partition of X generated by the orbits of our group action. [Answer: $\{\{(0,1,2),\ (1,2,0),\ (2,0,1)\},\ \{(0,2,1),\ (1,0,2),\ (2,1,0)\}\}$.]

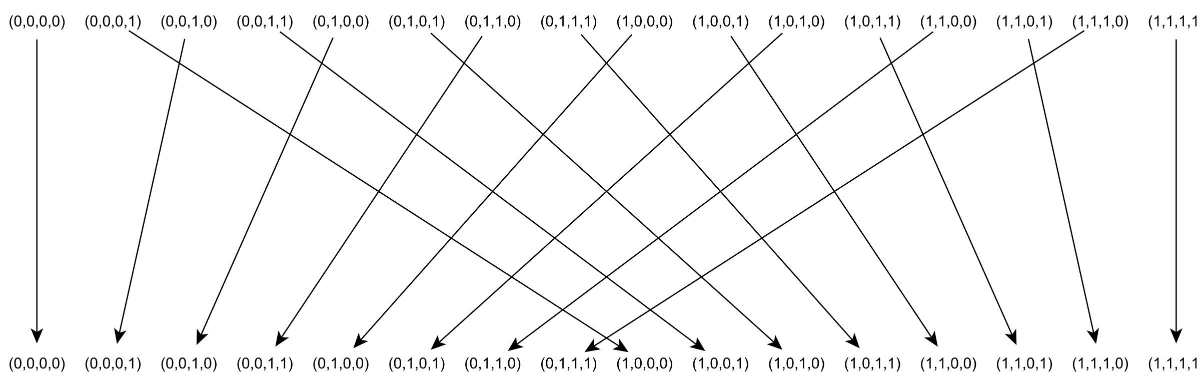
(c) Find the stabilizer of $(0,1,2)$. [Answer: The only element keeping $(0,1,2)$ fixed is the identity so $stab((0,1,2)) = \{e\}$.]

(d) Find the fixed points of $g = 1$. [Answer: As applying 1 alters each element we have $X^1 = \emptyset$.]

2. The following questions involve the $\mathbb{Z}_4$ group action on the set $(\mathbb{Z}_2)^4 = \{(a_1,a_2,a_3,a_4) : a_i \in \mathbb{Z}_2\}$ defined by $x \cdot (a_1,a_2,a_3,a_4) = (a_{1+x},a_{2+x},a_{3+x},a_{4+x})$ where the indices are computed modulo four with the smallest positive residue taken. Thus $0 \cdot (a,b,c,d) = (a,b,c,d)$, $1 \cdot (a,b,c,d) = (b,c,d,a)$, $2 \cdot (a,b,c,d) = (c,d,a,b)$, and $4 \cdot (a,b,c,d) = (d,a,b,c)$.

(a) Draw the permutation generated by $1 \in \mathbb{Z}_4$. [Answer: See figure 1.47]

Figure 1.47: The permutation generated by $1 \in \mathbb{Z}_4$ under this action on the set $(\mathbb{Z}_2)^4$.



(b) Draw the permutation generated by $2 \in \mathbb{Z}_4$. [Answer: See figure 1.48]

(c) Draw the permutation generated by $3 \in \mathbb{Z}_4$. [Answer: See figure 1.49]

(d) Find $orb((0,0,0,0))$. [Answer: $\{(0,0,0,0)\}$.]

(e) Find $orb((0,0,0,1))$. [Answer: $\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0)\}$.]

(f) Find $orb((0,0,1,1))$. [Answer: $\{(0,0,1,1),(0,1,1,0),(1,1,0,0),(1,0,0,1)\}$.]

(g) Find $orb((0,1,0,1))$. [Answer: $\{(0,1,0,1),(1,0,1,0)\}$.]

(h) Find $orb((0,1,1,1))$. [Answer: $\{(0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0)\}$.]

(i) Find $orb((1,1,1,1))$. [Answer: $\{(1,1,1,1)\}$.]

(j) Find the partition of X generated by the orbits of our group action. [Answer: $\{\{(0,0,0,0)\}$, $\{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0)\}$, $\{(0,0,1,1),(0,1,1,0),(1,1,0,0),(1,0,0,1)\}$, $\{(0,1,0,1),(1,0,1,0)\}$, $\{(0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0)\}$, $\{(1,1,1,1)\}\}$.]

Figure 1.48: The permutation generated by $2 \in \mathbb{Z}_4$ under this action on the set $(\mathbb{Z}_2)^4$.

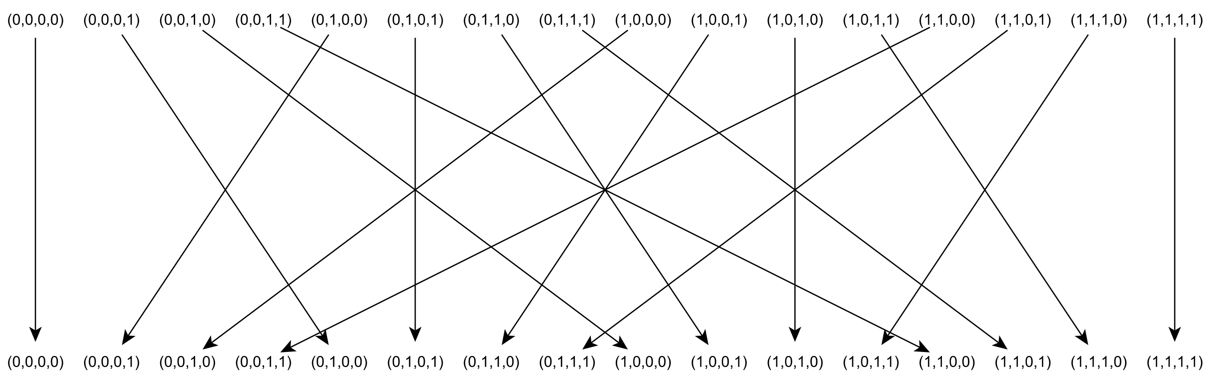
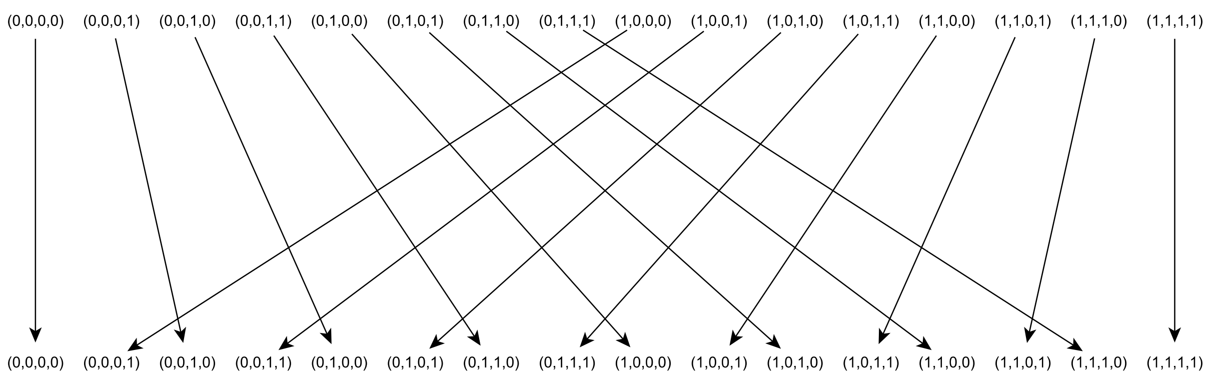


Figure 1.49: The permutation generated by $3 \in \mathbb{Z}_4$ under this action on the set $(\mathbb{Z}_2)^4$.



(k) Find $stab((0,0,0,0))$. [Answer: All of G.]

(l) Find $stab((0,0,0,1))$. [Answer: $\{0\}$.]

(m) Find $stab((0,1,0,1))$. [Answer: $\{0,2\}$.]

(n) Find $stab((1,0,0,1))$. [Answer: $\{0\}$.]

(o) Find X^0. [Answer: All of X.]

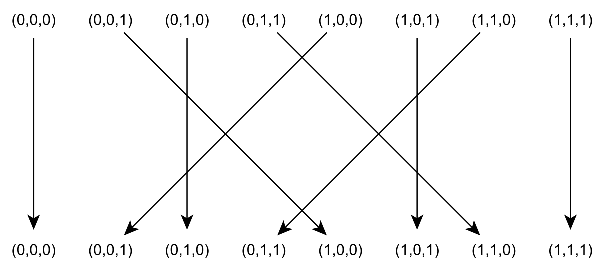
(p) Find X^1. [Answer: $\{(0,0,0,0),(1,1,1,1)\}$.]

(q) Find X^2. [Answer: $\{(0,0,0,0),(0,1,0,1),(1,0,1,0),(1,1,1,1)\}$.]

3. The following questions involve the $\mathbb{Z}_2$ group action on the set $(\mathbb{Z}_2)^3 = \{(a_1,a_2,a_3) : a_i \in \mathbb{Z}_2\}$ defined by setting $1 \cdot (a_1,a_2,a_3) = (a_3,a_2,a_1)$ and $0 \cdot (a_1,a_2,a_3) = (a_1,a_2,a_3)$.

   (a) Draw the permutation generated by $1 \in \mathbb{Z}_2$. [Answer: See figure 1.50]

Figure 1.50: The permutation generated by $1 \in \mathbb{Z}_2$ under this action on the set $(\mathbb{Z}_2)^3$.



(b) Find $orb((0,0,0))$. [Answer: $\{(0,0,0)\}$.]

(c) Find $orb((0,0,1))$. [Answer: $\{(0,0,1),(1,0,0)\}$.]

(d) Find $orb((0,1,0))$. [Answer: $\{(0,1,0)\}$.]

(e) Find $orb((0,1,1))$. [Answer: $\{(0,1,1),(1,1,0)\}$.]

(f) Find $orb((1,0,1))$. [Answer: $\{(1,0,1)\}$.]

(g) Find $orb((1,1,1))$. [Answer: $\{(1,1,1)\}$.]

(h) Find $stab((0,0,0))$. [Answer: All of G.]

(i) Find $stab((0,0,1))$. [Answer: $\{0\}$.]

(j) Find $stab((1,0,1))$. [Answer: All of G.]

(k) Find X^0. [Answer: All of X.]

(l) Find X^1. [Answer: $\{(0,0,0),(1,0,1),(0,1,0),(1,1,1)\}$.]

Chapter 2

The Orbit-Stabilizer Theorem

2.1 Matrices

1. Let X be the set of all two-by-two matrices with entries in $\mathbb{Z}_2$, and let $G = \mathbb{Z}_2 \cong \{e, t\}$ act on this set through transposition. Thus $t \cdot A = A^T$ and $e \cdot A = A$ for any matrix A in X.

   (a) For $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

   [Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{2}{2} = 1.$]

   (b) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

   [Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{2}{1} = 2.$]

   (c) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

   [Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{2}{1} = 2.$]

   (d) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

   [Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{2}{1} = 2.$]

   (e) For $x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

   [Answer: $stab(x) = \{e, t\}, |stab(x)| = 2, |orb(x)| = \frac{2}{2} = 1.$]

   (f) For $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

   [Answer: $stab(x) = \{e, t\}, |stab(x)| = 2, |orb(x)| = \frac{2}{2} = 1.$]

2. Let X be the set of all two-by-two matrices with entries in $\mathbb{Z}_2$, and let $G = \mathbb{Z}_2 \cong \{e, f\}$ act on this set

through reflection about a horizontal line of symmetry. Thus $f \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{2}{2} = 1.$]

(b) For $x = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{2}{1} = 2.$]

(c) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{2}{2} = 1.$]

(d) For $x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, f\}, |stab(x)| = 2, |orb(x)| = \frac{2}{2} = 1.$]

(e) For $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{2}{1} = 2.$]

3. Let X be the set of all two-by-two matrices with entries in $\mathbb{Z}_2$, and let $G = \mathbb{Z}_4 \cong \{e, r, r^2, r^3\}$ act on this set through rotation, with r rotating a matrix ninety degrees.

(a) For $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, |orb(x)| = 4, |stab(x)| = \frac{4}{4} = 1.$]

(b) For $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{4}{2} = 2.$]

(c) For $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{4}{1} = 4.$]

(d) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{4}{1} = 4.$]

(e) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, r^2\}, |stab(x)| = 2, |orb(x)| = \frac{4}{2} = 2$.]

(f) For $x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, r, r^2, r^3\}, |stab(x)| = 4, |orb(x)| = \frac{4}{4} = 1$.]

4. Let X be the set of all two-by-two matrices with entries in $\mathbb{Z}_2$, and let $G = D_4$ act on this set through rotation and reflection, with r rotating a matrix ninety degrees and f flipping a matrix over a vertical line of symmetry. For example, $r \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & a \\ c & b \end{bmatrix}$, $f \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$, and $rf \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$, and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, |orb(x)| = 4, |stab(x)| = \frac{8}{4} = 2$.]

(b) For $x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}, |orb(x)| = 4, |stab(x)| = \frac{8}{4} = 2$.]

(c) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{8}{2} = 4$.]

(d) For $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{8}{1} = 8$.]

(e) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, rf\}, |stab(x)| = 2, |orb(x)| = \frac{8}{2} = 4$.]

(f) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, r^3 f\}, |stab(x)| = 2, |orb(x)| = \frac{8}{2} = 4$.]

(g) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, r^2 f\}, |stab(x)| = 2, |orb(x)| = \frac{8}{2} = 4$.]

(h) For $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, r^2, rf, r^3 f\}, |stab(x)| = 4, |orb(x)| = \frac{8}{4} = 2$.]

(i) For $x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = G, |stab(x)| = 8, |orb(x)| = \frac{8}{8} = 1.$]

(j) Is it possible for $orb(x)$ to equal X?
[Answer: No. That requires the stabilizer to equal the identity, and that is impossible for a two-by-two binary matrix. A matrix with one or three ones has diagonal symmetry and thus rf or r^3f is in the stabilizer. A matrix with two zeroes has diagonal symmetry if they are on the diagonal, and horizonal or vertical symmetry otherwise. Finally, a matrix with no or four zeroes has stabilizer G.]

5. Let X be the set of all two-by-two invertible matrices with entries in $\mathbb{Z}_2$, and let $G = \mathbb{Z}_2 \cong \{e, i\}$ act on this set through inversion. For example, $X = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$, $i \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $i \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{2}{1} = 2.$]

(b) For $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}, |orb(x)| = 2, |stab(x)| = \frac{2}{2} = 1.$]

(c) For $x = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{2}{1} = 2.$]

(d) For $x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e, i\}, |stab(x)| = 2, |orb(x)| = \frac{2}{2} = 1.$]

6. Let X be the set of all three-by-three matrices with entries in $\mathbb{Z}_2$, and let $G = \mathbb{Z}_3 \cong \{e, a, a^2\}$ act on this set by permuting the columns of each matrix. Let $a \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} c & a & b \\ f & d & e \\ i & g & h \end{bmatrix}$, $a^2 \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} b & c & a \\ e & f & d \\ h & i & g \end{bmatrix}$, and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}, |orb(x)| = 3, |stab(x)| = \frac{3}{3} = 1.$]

(b) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, find $orb(x)$, $|orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right\}$, $|orb(x)| = 1, |stab(x)| = \frac{3}{1} = 3$.]

(c) For $x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, find $orb(x)$, $|orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$, $|orb(x)| = 3, |stab(x)| = \frac{3}{3} = 1$.]

(d) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, a, a^2\}, |stab(x)| = 3, |orb(x)| = \frac{3}{3} = 3$.]

7. Let X be the set of all three-by-three matrices with entries in $\mathbb{Z}_2$, and let $G = S_3$ act on this set by permuting the columns of each matrix. For example $(1,2,3) \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} c & a & b \\ f & d & e \\ i & g & h \end{bmatrix}$, and $(1,2) \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}$, and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $orb(x)$, $|orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$, $|orb(x)| = 3, |stab(x)| = \frac{6}{3} = 2$.]

(b) For $x = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, find $orb(x)$, $|orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \right\}$,
$|orb(x)| = 6, |stab(x)| = \frac{6}{6} = 1$.]

(c) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, (1,3)\}, |stab(x)| = 2, |orb(x)| = \frac{6}{2} = 3$.]

(d) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = S_3, |stab(x)| = 6, |orb(x)| = \frac{6}{6} = 1$.]

(e) For $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find $stab(x), |stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{6}{1} = 6$.]

8. Let X be the set of all three-by-three matrices with entries in $\mathbb{Z}_2$, and let $G = D_4$ act on this set by ninety degree rotation and reflection about a vertical line of symmetry. For example $r \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & d & a \\ h & e & b \\ i & f & c \end{bmatrix}$, and $f \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} c & b & a \\ f & e & d \\ i & h & g \end{bmatrix}$, and $e \cdot A = A$ for any $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ in X.

(a) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 4, |stab(x)| = \frac{8}{4} = 2$.]

(b) For $x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}, |orb(x)| = 1, |stab(x)| = \frac{8}{1} = 8$.]

(c) For $x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right\}, |orb(x)| = 4, |stab(x)| = \frac{8}{4} = 2$.]

(d) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find $orb(x), |orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.
[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$
$8, |stab(x)| = \frac{8}{8} = 1$.]

(e) For $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find $orb(x)$, $|orb(x)|$, and use $|orb(x)|$ to find $|stab(x)|$.

[Answer: $orb(x) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$, $|orb(x)| = 2, |stab(x)| = \frac{8}{2} = 4$.]

(f) For $x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, rf\}, |stab(x)| = 2, |orb(x)| = \frac{8}{2} = 4$.]

(g) For $x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e\}, |stab(x)| = 1, |orb(x)| = \frac{8}{1} = 8$.]

(h) For $x = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, r^2, rf, r^3f\}, |stab(x)| = 4, |orb(x)| = \frac{8}{4} = 2$.]

(i) For $x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = D_4, |stab(x)| = 8, |orb(x)| = \frac{8}{8} = 1$.]

(j) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, r^2, f, r^2f\}, |stab(x)| = 4, |orb(x)| = \frac{8}{4} = 2$.]

(k) For $x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, find $stab(x)$, $|stab(x)|$, and use $|stab(x)|$ to find $|orb(x)|$.

[Answer: $stab(x) = \{e, f\}, |stab(x)| = 2, |orb(x)| = \frac{8}{2} = 4$.]

2.2 Necklaces

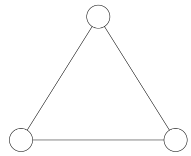
1. Let X be the set of all three beaded necklaces with blue and white beads. Allow the group D_3 to act upon that set by rotations and reflections. Specifically, the element f flips the necklace over a vertical line of symmetry, and the element r rotates the element clockwise 120 degrees.

   (a) For the following questions, let x be the necklace shown in figure 2.1.

      i. Find $stab(x)$.
      [Answer: All of D_3.]

      ii. Use $|stab(x)|$ to find $|orb(x)|$.
      [Answer: Since $|stab(x)| = 6$, $|orb(x)| = \frac{6}{6} = 1$.]

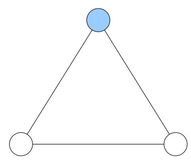
Figure 2.1: A necklace with all white beads.



iii. Find $orb(x)$
[Answer: The orbit is just $\{x\}$.]

(b) For the following questions, let x be the necklace shown in figure 2.2.

Figure 2.2: A necklace with one blue bead.

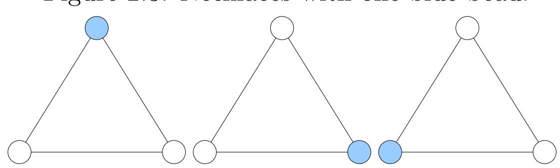


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{6}{2} = 3$.]

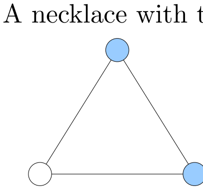
iii. Find $orb(x)$
[Answer: The orbit is the set of the three necklaces shown in figure 2.3.]

Figure 2.3: Necklaces with one blue bead.



(c) For the following questions, let x be the necklace shown in figure 2.4.

Figure 2.4: A necklace with two blue beads.

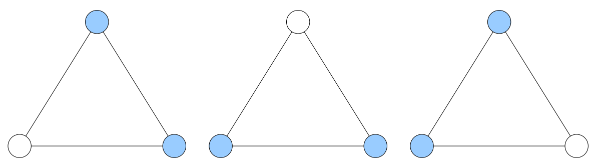


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, rf\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{6}{2} = 3$.]

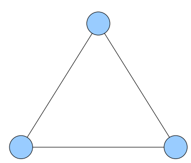
iii. Find $orb(x)$
[Answer: The orbit is the set of the three necklaces shown in figure 2.5.]

Figure 2.5: Necklaces with two blue beads.



(d) For the following questions, let x be the necklace shown in figure 2.6.

Figure 2.6: A necklace with all blue beads.



i. Find $stab(x)$.
[Answer: All of D_3.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 6$, $|orb(x)| = \frac{6}{6} = 1$.]

iii. Find $orb(x)$
[Answer: The orbit is just $\{x\}$.]

2. Let X be the set of all three beaded necklaces with dark red, blue and white beads. Allow the group D_3 to act upon that set by rotations and reflections. Specifically, the element f flips the necklace over a vertical line of symmetry, and the element r rotates the element clockwise 120 degrees. For the following questions, let x be the necklace shown in figure 2.7.

(a) Find $stab(x)$.
[Answer: $stab(x) = \{e, rf\}$.]

(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{6}{1} = 6$.]

(c) Find $orb(x)$
[Answer: The orbit is the set of the six necklaces shown in figure 2.8.]

Figure 2.7: A necklace with one red, one blue, and one white bead.

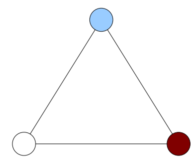
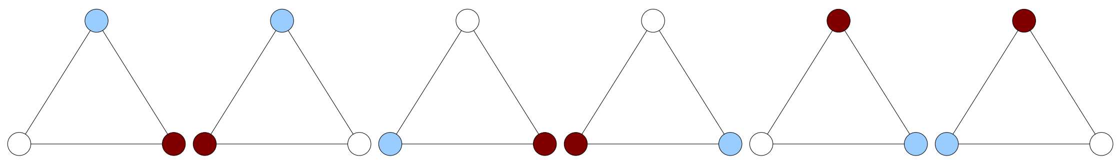


Figure 2.8: Necklaces with one red, one blue, and one white bead.



3. Let X be the set of all three beaded necklaces with dark red, blue and white beads. Allow the group $\mathbb{Z}_3 \cong \{e, r, r^2\}$ to act upon that set by rotations and reflections. Specifically, the element r rotates the element clockwise 120 degrees.

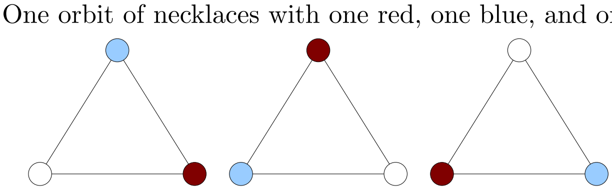
   (a) For the following questions, let x be the necklace shown in figure 2.7.

      i. Find $stab(x)$.
      [Answer: $stab(x) = \{e\}$.]

      ii. Use $|stab(x)|$ to find $|orb(x)|$.
      [Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{3}{1} = 3$.]

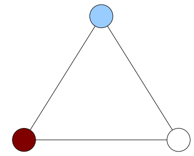
      iii. Find $orb(x)$
      [Answer: The orbit is the set of the three necklaces shown in figure 2.9.]

Figure 2.9: One orbit of necklaces with one red, one blue, and one white bead.



   (b) For the following questions, let x be the necklace shown in figure 2.10.

Figure 2.10: A necklace with one red, one blue, and one white bead.

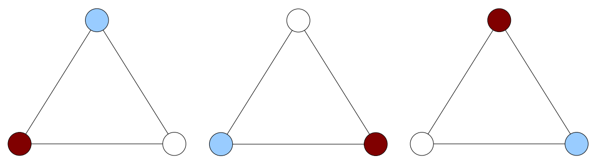


i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{3}{1} = 3$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the three necklaces shown in figure 2.11.]

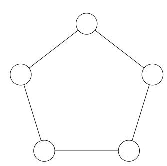
Figure 2.11: Another orbit of necklaces with one red, one blue, and one white bead.



4. Let X be the set of all five beaded necklaces with blue and white beads. Allow the group D_5 to act upon that set by rotations and reflections. Specifically, the element f flips the necklace over a vertical line of symmetry, and the element r rotates the element clockwise 120 degrees.

   (a) For the following questions, let x be the necklace shown in figure 2.12.

Figure 2.12: A necklace with five white beads.



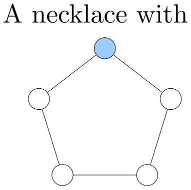
i. Find $stab(x)$.
[Answer: All of D_5.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 10$, $|orb(x)| = \frac{10}{10} = 1$.]

iii. Find $orb(x)$
[Answer: The orbit is just $\{x\}$.]

   (b) For the following questions, let x be the necklace shown in figure 2.13.

Figure 2.13: A necklace with one blue bead.

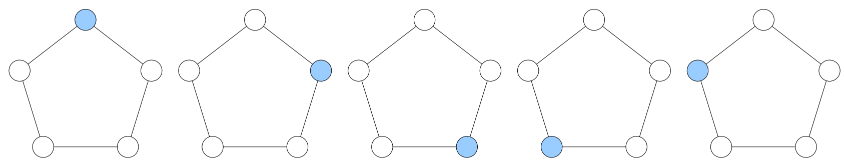


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{10}{2} = 5$.]

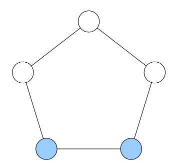
iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.14.]

Figure 2.14: Necklaces with one blue bead.



(c) For the following questions, let x be the necklace shown in figure 2.15.

Figure 2.15: A necklace with two blue beads.

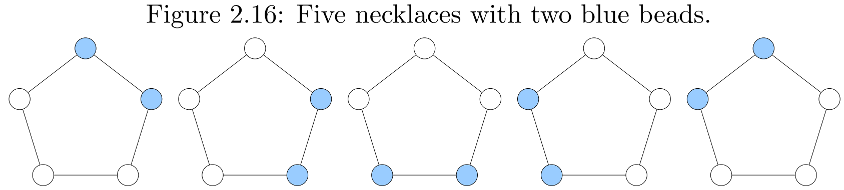


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{10}{2} = 5$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.16.]

Figure 2.16: Five necklaces with two blue beads.

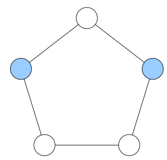


(d) For the following questions, let x be the necklace shown in figure 2.17.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

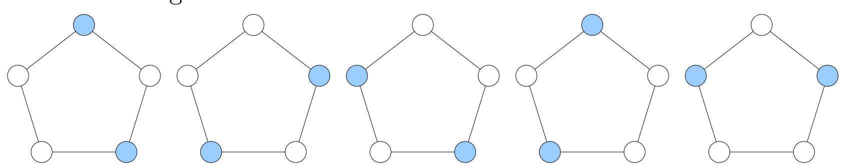
ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{10}{2} = 5$.]

Figure 2.17: A necklace with two blue beads.



iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.18.]

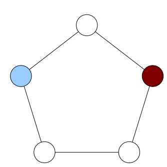
Figure 2.18: Five necklaces with two blue beads.



5. Let X be the set of all five beaded necklaces with dark red, blue and white beads. Allow the group D_5 to act upon that set by rotations and reflections. Specifically, the element f flips the necklace over a vertical line of symmetry, and the element r rotates the element clockwise 120 degrees.

   (a) For the following questions, let x be the necklace shown in figure 2.19.

Figure 2.19: A necklace with one red and one blue bead.



i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{10}{1} = 10$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the ten necklaces shown in figure 2.20.]

   (b) For the following questions, let x be the necklace shown in figure 2.21.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{10}{2} = 5$.]

Figure 2.20: Ten necklaces with one red and one blue bead.

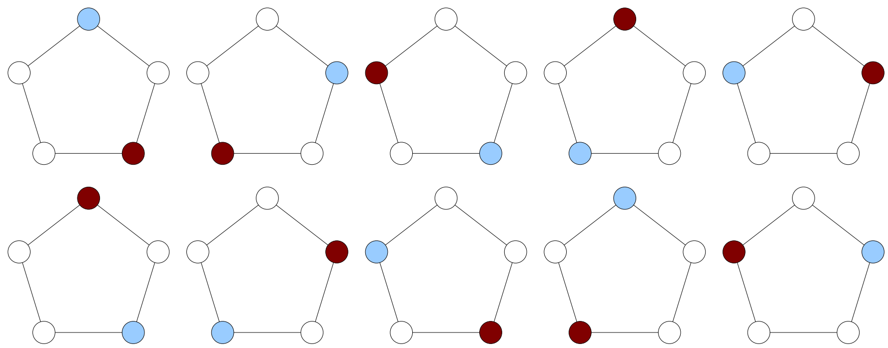
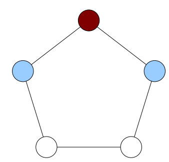
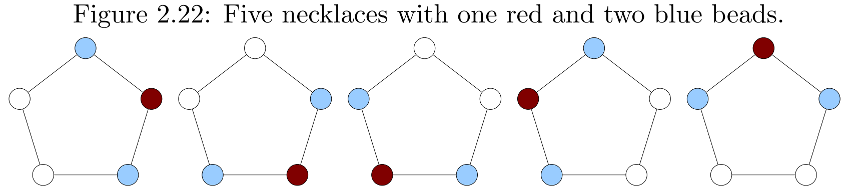


Figure 2.21: A necklace with one red and two blue beads.



iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.22.]

Figure 2.22: Five necklaces with one red and two blue beads.



6. Let X be the set of all five beaded necklaces with dark red, blue and white beads. Allow the group $\mathbb{Z}_5$ to act upon that set by rotations and reflections. Specifically, the element r rotates the element clockwise 120 degrees.

(a) For the following questions, let x be the necklace shown in figure 2.19.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{5}{1} = 5$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.23.]

Figure 2.23: Ten necklaces with one red and one blue bead.

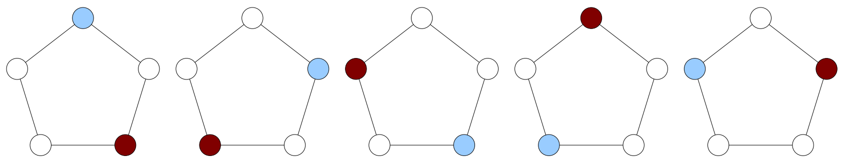
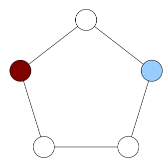


Figure 2.24: A necklace with one red and one blue bead.



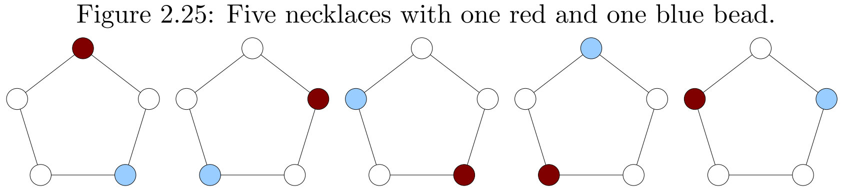
(b) For the following questions, let x be the necklace shown in figure 2.24.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{5}{1} = 5$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.25.]

Figure 2.25: Five necklaces with one red and one blue bead.



7. Let X be the set of all six beaded necklaces with dark red, blue and white beads. Allow the group D_6 to act upon that set by rotations and reflections. Specifically, the element f flips the necklace over a vertical line of symmetry, and the element r rotates the element clockwise 120 degrees.

(a) For the following questions, let x be the necklace shown in figure 2.26.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{12}{2} = 6$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of six necklaces shown in figure 2.27.]

(b) For the following questions, let x be the necklace shown in figure 2.28.

Figure 2.26: A necklace with two blue beads.

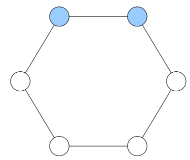


Figure 2.27: Six necklaces with two blue beads.

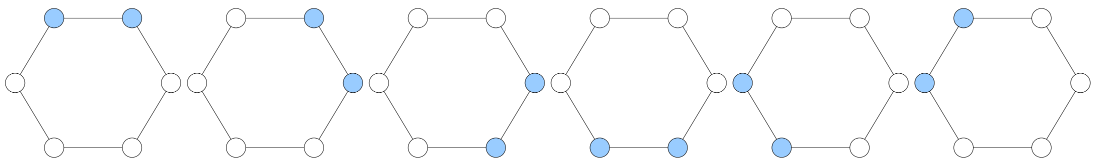
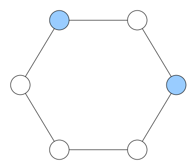


Figure 2.28: A necklace with two blue beads.

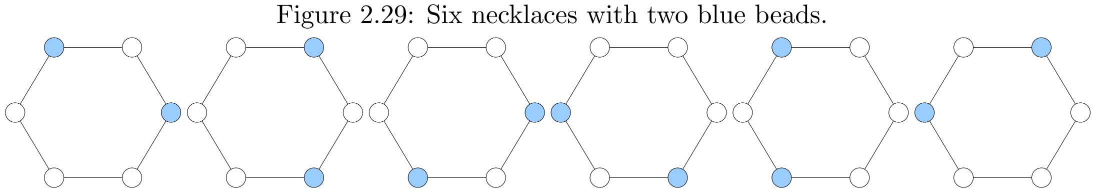


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, rf\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{12}{2} = 6$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of the five necklaces shown in figure 2.29.]

Figure 2.29: Six necklaces with two blue beads.

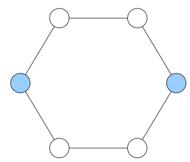


(c) For the following questions, let x be the necklace shown in figure 2.30.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e, f, r^3, r^3f\}$.]

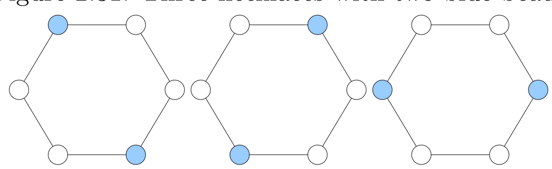
ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 4$, $|orb(x)| = \frac{12}{4} = 3$.]

Figure 2.30: A necklace with two blue beads.



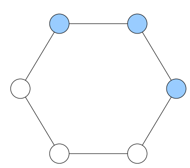
iii. Find $orb(x)$
[Answer: The orbit is the set of three necklaces shown in figure 2.31.]

Figure 2.31: Three necklaces with two blue beads.



(d) For the following questions, let x be the necklace shown in figure 2.32.

Figure 2.32: A necklace with three blue beads.

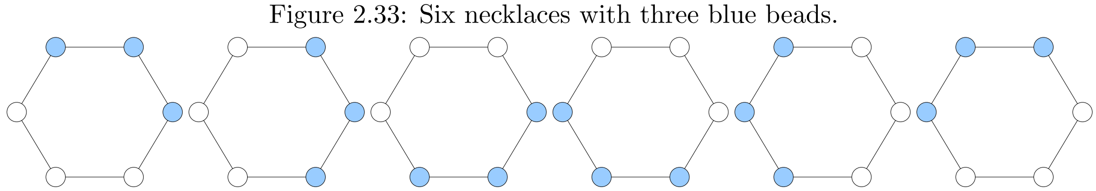


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, rf\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{12}{2} = 6$.]

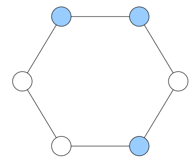
iii. Find $orb(x)$
[Answer: The orbit is the set of six necklaces shown in figure 2.33.]

Figure 2.33: Six necklaces with three blue beads.



(e) For the following questions, let x be the necklace shown in figure 2.34.

Figure 2.34: A necklace with three blue beads.

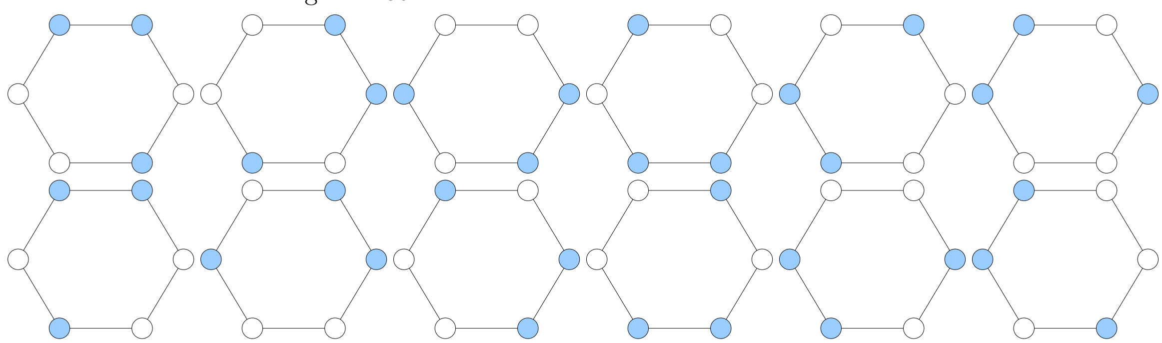


i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{12}{1} = 12$.]

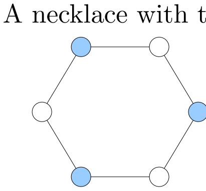
iii. Find $orb(x)$
[Answer: The orbit is the set of twelve necklaces shown in figure 2.35.]

Figure 2.35: Twelve necklaces with three blue beads.



(f) For the following questions, let x be the necklace shown in figure 2.36.

Figure 2.36: A necklace with three blue beads.



i. Find $stab(x)$.
[Answer: $stab(x) = \{e, r^2, r^3, rf, r^3f, r^5f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 6$, $|orb(x)| = \frac{12}{6} = 2$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of two necklaces shown in figure 2.37.]

Figure 2.37: Two necklaces with three blue beads.

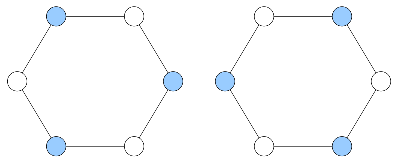
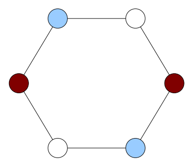


Figure 2.38: A necklace with two white, two blue, and two red beads.



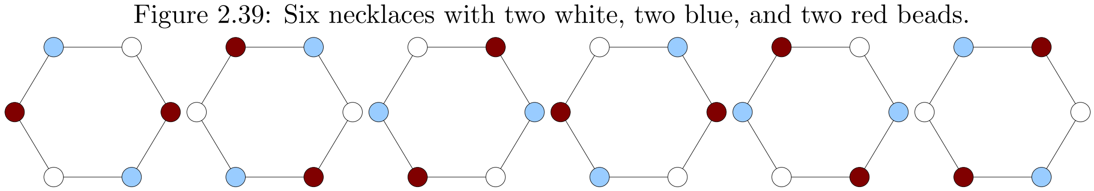
(g) For the following questions, let x be the necklace shown in figure 2.38.

i. Find $stab(x)$.
[Answer: $stab(x) = \{e, r^3\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{12}{2} = 6$.]

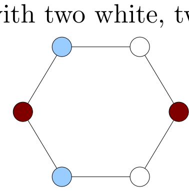
iii. Find $orb(x)$
[Answer: The orbit is the set of six necklaces shown in figure 2.39.]

Figure 2.39: Six necklaces with two white, two blue, and two red beads.



(h) For the following questions, let x be the necklace shown in figure 2.40.

Figure 2.40: A necklace with two white, two blue, and two red beads.

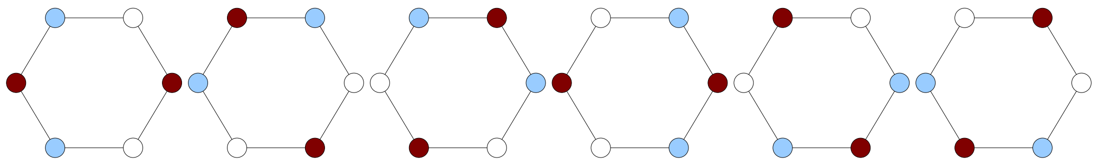


i. Find $stab(x)$.
[Answer: $stab(x) = \{e, r^3 f\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 2$, $|orb(x)| = \frac{12}{2} = 6$.]

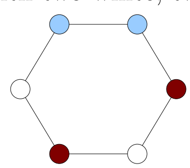
iii. Find $orb(x)$
[Answer: The orbit is the set of six necklaces shown in figure 2.41.]

Figure 2.41: Six necklaces with two white, two blue, and two red beads.



(i) For the following questions, let x be the necklace shown in figure 2.42.

Figure 2.42: A necklace with two white, two blue, and two red beads.

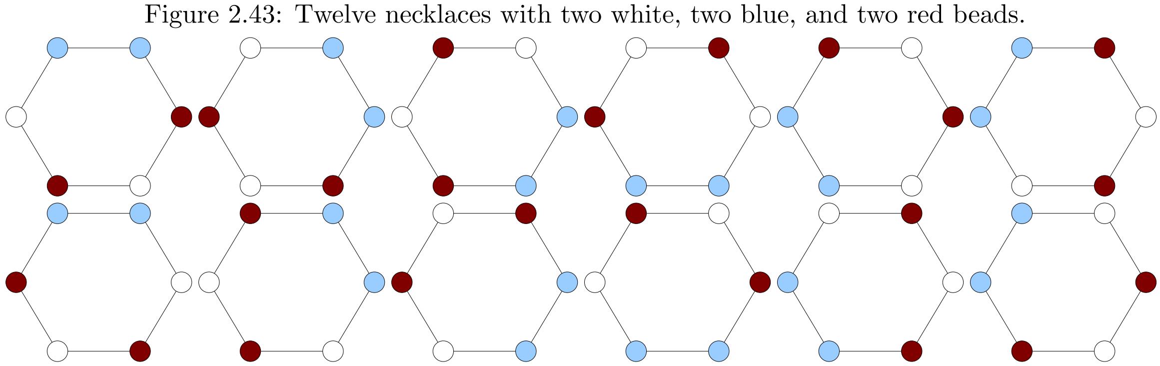


i. Find $stab(x)$.
[Answer: $stab(x) = \{e\}$.]

ii. Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: Since $|stab(x)| = 1$, $|orb(x)| = \frac{12}{1} = 12$.]

iii. Find $orb(x)$
[Answer: The orbit is the set of twelve necklaces shown in figure 2.43.]

Figure 2.43: Twelve necklaces with two white, two blue, and two red beads.



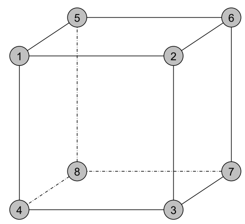
2.3 Platonic Solids

The following questions involve platonic solids and the Orbit-Stabilizer Theorem.

2.3.1 The Cube

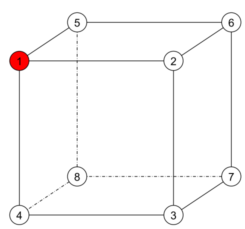
For the following questions let G be the group of rotational symmetries of this cube. This is the group $G = S_4$ of rotations in 3-space that leave the sides, corners and edges facing the same directions.

Figure 2.44: A picture of a cube.



1. Consider the set of cubes with red and white corners under the action of the group of rotational symmetries of the cube.
   (a) Find $|orb_G(x)|$ where x is the cube with exactly one red corner shown in figure 2.45. All other corners are white.
   [Answer: We can map the red corner to any of the corners so there are eight possibilities. Thus $|orb_G(x)| = 8$.]

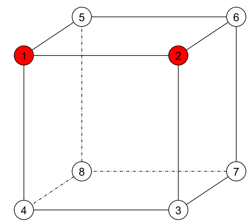
Figure 2.45: A cube with one red corner.



   (b) Find $|stab_G(x)|$ where x is the cube in figure 2.45. What is $stab_G(x)$ isomorphic to?
   [Answer: As that corner must stay in place, the only motions that fix this must be combinations of 120 degree rotations (also $\tau/3$ or $2\pi/3$) around the diagonal going through corner one and corner seven. Thus $|stab_G(x)| = 3$. This is generated by one 120 degree twist, therefore cyclic and isomorphic to $\mathbb{Z}_3$. Alternatively, if we already know $|G| = 24$ then we can use the orbit stabilizer theorem to get that the size must be three. Any group of order three is cyclic and therefore this is isomorphic to $\mathbb{Z}_3$.]
   (c) Find $|stab_G(x)|$ where x is the cube with two red corners shown in figure 2.46. What is $stab_G(x)$ isomorphic to?

[Answer: The motions that fix this must rotate the cube about the midpoints of edge $\{1, 2\}$ and $\{7, 8\}$ 180 degrees (also π or $\tau/2$.) Thus we get $|stab_G(x)| = 2$ and that $stab_G(x) = \mathbb{Z}_2$.]

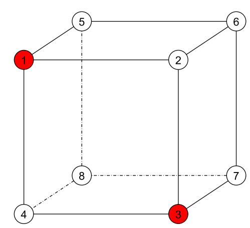
Figure 2.46: A cube with two red corners.



(d) Find $|orb_G(x)|$ where x is the cube with two red corners shown in figure 2.46.
[Answer: There are eight corners we can color red, and each has exactly three neighbors that we can color red as well. In doing this, we will "accidentally" count each combination twice so we must divide by two when we are done. This gives us $8 \cdot 3/2 = 12$ possibilities. Alternatively we can use the orbit stabilizer theorem if we already know $|stab_G(x)|$ to arrive at the same result.]

(e) Find $|stab_G(x)|$ where x is the cube with two red corners shown in figure 2.47. What is $stab_G(x)$ isomorphic to?
[Answer: The motions that fix this must rotate the $\{1, 2, 3, 4\}$ edge 180 degrees (also π or $\tau/2$.) Thus we get $|stab_G(x)| = 2$ and that $stab_G(x) = \mathbb{Z}_2$.]

Figure 2.47: Another cube with two red corners.

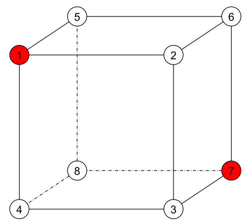


(f) Find $|orb_G(x)|$ where x is the cube with two red corners shown in figure 2.47.
[Answer: There are six sides. Each has two pairs of opposite corners. This gives twelve possibilities in total. Alternatively, there are eight corners we can color red, each touching three sides where we can add another red at the opposite corner. This counts each possibility twice so we get $8 \cdot 3/2 = 12$ possibilities. We can also use the orbit stabilizer theorem.]

(g) Find $|stab_G(x)|$ where x is the cube with two red corners shown in figure 2.48. What is $stab_G(x)$ isomorphic to?
[Answer: Any 120 degree rotation along the 1-7 axis will fix this. We can also switch corners 1 and 7 and apply any 120 dgree rotation. Thus we get a copy of D_3 which preserves x and $|stab_G(x) \cong 6|$.]

(h) Find $|orb_G(x)|$ where x is the cube with two red corners shown in figure 2.48.
[Answer: When colored red each corner has only one opposite corner we can color red. There are

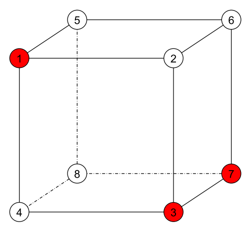
Figure 2.48: A cube with two red corners.



8 corners, but this method will count each possibility twice, so we get a total of four possibilities. Of course, the orbit stabilizer theorem gives us this as well.]

(i) Find $|orb_G(x)|$ and $|stab_G(x)|$ where x is the cube shown in figure 2.49. State what $stab_G(x)$ is isomorphic to.
[Answer: We can pick any two opposite corners in four different ways. Each of those has six ways to color one adjacent corner. This gives $|orb_G(x)| = 24$. Alternatively, as $|stab_G(x)| = 1$ we know $|orb_G(x)| = 24$ as well. $stab_G(x)$ is the identity group.]

Figure 2.49: A cube with three red corners with two reds in opposite corners.



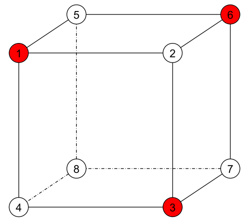
(j) Can the orbit of any configuration of any cube with red and white colored corners be more than 24?
[Answer: No, as that would imply a fractional stabilizer.]

(k) Can the orbit of any configuration of a cube with corners colored in any number of colors be more than 24?
[Answer: No. Again that would imply a fractional stabilizer.]

(l) Find $|orb_G(x)|$ and $|stab_G(x)|$ where x is the cube shown in figure 2.50. State what $stab_G(x)$ is isomorphic to.
[Answer: We can pick any one corner, leave that white, and color the three adjacent corners. We can do this in eight ways so $|orb_G(x)| = 8$. The stabilizer if given by rotating through the axis of this white corner surrounded by red corners and the opposite corner by 120 degrees. This gives $|orb_G(x)| = 3$ and $stab_G(x)$ isomorphic to $\mathbb{Z}_3$.]

(m) Does every coloring with three red corners and five white corners fall into the orbits of figure 2.49 and figure 2.50?
[Answer: The orbits in a group action always partition the set acted upon. Here, our action is on the set of all cubes with three red and five white corners. There are eight choose three different

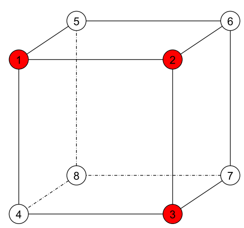
Figure 2.50: A cube with three red corners and no adjacent red corners.



ways to color three corners, which gives fifty-six possibilities in our base set. The orbit of figure 2.49 only has 24 possibilities, and the orbit of figure 2.50 only had 8 possibilities. There are 32 possibilities in total in these two orbits and 24 elements in our set do not fall into these orbits.]

(n) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.51. State what $stab_G(x)$ is isomorphic to.
[Answer: Note that all three vertices are along exactly one face of the cube. We can pick a face in six ways, then pick which corner not to color in four ways. This gives $6 \cdot 4 = 24$ possibilities and thus $|orb_G(x)| = 24$. Another way to do this is to notice that as the three are in a row, there is always one "middle" red corner. We can pick which corner is middle in one of eight ways. Once the middle is chose, there are three choices for which adjacent corners we pick. This gives $8 \cdot 3$ possibilities, showing $|orb_G(x)| = 24$. We can also get the orbit from the fact that the stabilizer is just the identity group. This shows $|stab_G(x)| = 1$ and thus $|orb_G(x)| = |G|/|stab_G(x)| = 24$.]

Figure 2.51: A cube with three red corners where one red corner has two red neighbors.



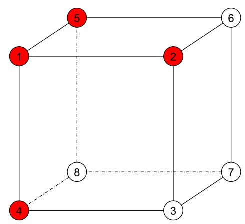
(o) How many elements fall into the orbits of figure 2.49, figure 2.50, and figure 2.51 in this action on the set of cubes with three red and five white corners? How many elements do not?
[Answer: These orbits are disjoint and have 24, 8 and 24 elements respectively. This means every possibility falls into one of these three cases.]

(p) Explain without the language of group actions why all possibilities for cubes with three red and five white corners can be rotated to look like figure 2.49, figure 2.50, or figure 2.51.
[Answer: If two opposite corners are red, then the third must touch exactly one, leaving us in the case of figure 2.49, regardless of which. Remember, we can rotate so the one with a red companion is in spot seven and spin along the one-seven axis till the companion is in spot three. We need only count the other possibilities two where there aren't reds in opposite corners. We have the case where no reds are adjacent, which gives us figure 2.50. We can't have two adjacent and third

third one not without putting a red in one of the two opposite corners. Thus all three reds have at least one adjacent neighbor. This requires a red path as in figure 2.51.]

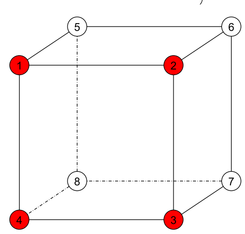
(q) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.52. State what $stab_G(x)$ is isomorphic to.
[Answer: Here there is one corner with three red neighbors. We can pick that corner in eight ways, and each of those ways determines the coloring. Thus we have $|orb_G(x)| = 8$. This implies the stabilizer has order three, and is isomorphic to $\mathbb{Z}_3$. We also can see this as it must equal the group of rotations along the one-seven axis by multiples of 120 degrees.]

Figure 2.52: A cube with three red corners where one red corner has three red neighbors.



(r) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.53. State what $stab_G(x)$ is isomorphic to.
[Answer: The $\{1, 2, 3, 4\}$ face must stay red, so this means the cyclic group of order four generated by turning the face ninety degrees is the stabilizer group. As $|stab_G(x)| = 4$ we know $|orb_G(x)| = 6$. We can also see this because coloring the edges of any one face can be done in six ways.]

Figure 2.53: A cube with four red corners, each with two red neighbors.



(s) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.54. State what $stab_G(x)$ is isomorphic to.
[Answer:Pick any edge and color both corners. Then color corners adjacent to those without making one side red. This can be done in two ways once a corner is picked. Only one of those ways can be rotated to look like figure 2.54. The other looks like figure 2.55. Thus once we know which edge gets two colored vertices, the cube is determined. This gives us $|orb_G(x)| = 12$. We can also get this from the stabilizer, which is equal to $\mathbb{Z}_2$ because the only nontrivial motion preserving this figure is the 180 degree rotation about the midpoints of the $\{1, 2\}$ and $\{7, 8\}$ edges.]

(t) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.56. State what $stab_G(x)$ is isomorphic to.

Figure 2.54: A cube with four red corners in a path.

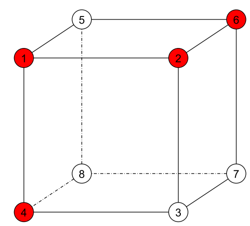
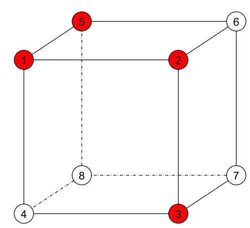
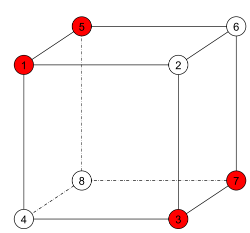


Figure 2.55: A different cube with four red corners in a path.



[Answer: Four our stabilizer, notice that we can rotate about the midpoint of the $\{1, 5\}$ and $\{3, 7\}$ edges by 180 degrees, rotate the $\{1, 2, 3, 4\}$ face 180 degrees, or do both. Thinking of one of these as a "flip", and the other a "rotate" (it doesn't matter which) we get a a copy of the dihedral group D_2 which is equal to the Klein-four group V. As the stabilizer has order four the orbit has size six. We can also get this by coloring the corners of any two opposite edges. As there are twelve edges, there are exactly six ways we can do this, which gives us $|orb_G(x)| = 6$.]

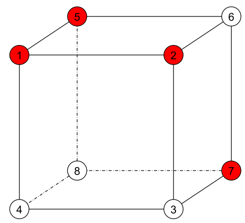
Figure 2.56: A cube with four red corners.



(u) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.57. State what $stab_G(x)$ is isomorphic to.
[Answer: The stabilizer here is just the identity and has order one. Corners one and seven are distinguished and must stay in place, and any rotation about the one-seven axis leads to a different configuration. This means the orbit must have size twenty-four. We can also get this by picking the distinguished "lonely" red corner in one of eight ways and then choosing the opposite corner and any two of its neighbors in three choose two ways. This also gives us $|orb_G(x)| = 8 \cdot 3 = 24$.]

(v) How many cubes with four red and four white corners fall into the orbits of figure 2.52 through

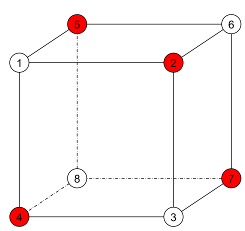
Figure 2.57: A cube with four red corners without much symmetry.



2.57? How many do not?
[Answer: All these orbits are distinct, so each cube can only fall into one orbit. We see the orbits have sizes 8,6,12,12,6 and 24 which sums to sixty-eight. For the second question, note that eight choose four is seventy, so there must be exactly two cubes that are not rotationally equivalent to the ones in figures 2.52 through 2.57. Thus there are either two orbits of size one left, or one orbit of size two left.]

(w) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.58. State what $stab_G(x)$ is isomorphic to.
[Answer: Picture the tetrahedron inside this cube with corners equal to the red corners of the cube. Any rotational symmetry of this tetrahedron preserves the coloring of this cube. This group is A_4. The order of the group is twelve, thus the $stab_G(x)$ is at least twelve. By Lagrange, the stabilizer has size either 12 or 24 since it divides 24. Now the stabilizer is not everything, as rotating an edge ninety degrees does not preserve the coloring. Thus $stab_G(x)$ must be A_4 and has order twelve. This implies the orbit has order two. We can find the orbit a different way. After coloring a corner red, any color an even number of steps away must also be red. This means we can complete the coloring in only one way. Four of the corners result in the picture shown and the other four result in the opposite coloring. Thus the orbit has size two.]

Figure 2.58: A cube with four red corners with much symmetry.



(x) How many orbits are on the collection of cubes with four red and four white corners, under the action of the rotational symmetries of the cube? How many distinct ways can we color the corners of a cube that we can rotate however we like if we want exactly four red, and four white corners?
[Answer: Everything falls into one of the seven orbits shown. The second question here is identical to the first, thus the answer is seven.]

(y) How many distinct ways can we color the corners of a cube that we can rotate however we like using two colors?

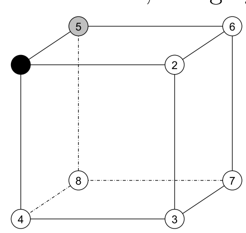
[Answer: Here we can use the previous answers along the the fact that the number of ways to color the cube with n red and $8-n$ white corners is the same as the number of ways to color the cube with $8-n$ red and n white corners. To see this, simply think of the bijection that switches the colors of each corner. We have one way to color something with no red corners. Our previous problems show there is one orbit in the one red case, three orbits in the two red and six white case, three orbits in the three red and five white case, and seven orbits in the four of each case. Reversing colors gives us three orbits for the five red case, three for the six red case, and one orbit each for the seven red and eight red cases. Thus we have 1+1+3+3+7+3+3+1+1=23 possible colorings.]

2. Consider the set X of cubes with one black, one grey and six white corners. Allow the group of rotational symmetries of the cube to act upon this set.

   (a) What is the size of X?
   [Answer: We have eight ways to pick one black corner and then seven remaining ways to pick one grey one. Thus there are a total of $8 \times 7 = 56$ possibilities.]

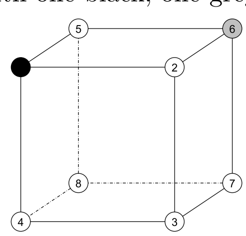
   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.59.
   [Answer: $|orb_G(x)| = 24$, and $|stab_G(x)| = 1$.]

Figure 2.59: A cube with one black, one grey, and six white corners.



   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.60.
   [Answer: $|orb_G(x)| = 24$, and $|stab_G(x)| = 1$.]

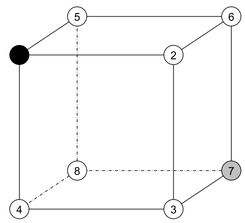
Figure 2.60: A cube with one black, one grey, and six white corners.



   (d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.61.
   [Answer: $|orb_G(x)| = 8$, and $|stab_G(x)| = 3$.]

   (e) Can every cube with one black, one grey and six white corners be rotated to look like one of the cubes in figures 2.59 through 2.61?

Figure 2.61: A cube with one black, one grey, and six white corners.

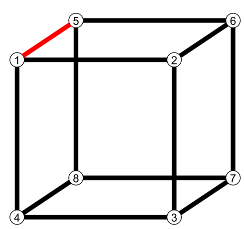


[Answer: Their orbits are disjoint and contain 24, 24 and 8 elements respectively. Together this gives us 56 cubes. Since $|X| = 56$, every cube must fall into one of these orbits.]

3. Consider the set X of cubes with red and black edges under the action of the group of rotational symmetries of the cube.

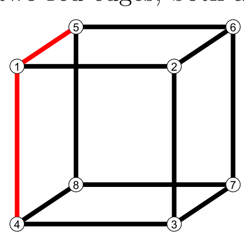
   (a) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.62.
   [Answer: The group of rotational symmetries takes this red edge to any edge, thus $|orb_G(x)| = 12$ implying $|stab_G(x)| = 2$ and $stab_G(x) \cong \mathbb{Z}_2$. Alternately, to fix the red edge where it is, we need to rotate about the midpoint of the $\{1, 5\}$ and $\{3, 7\}$ edges, which also imples $stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.62: A cube with one red edge.



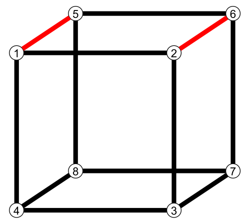
   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.63.
   [Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.63: A cube with two red edges, both adjacent to the same corner.



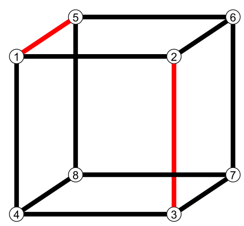
   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.64.
   [Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.64: A cube with two red edges.



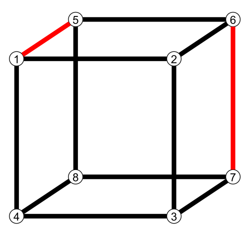
(d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.65.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.65: A cube with two red edges.



(e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.66.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

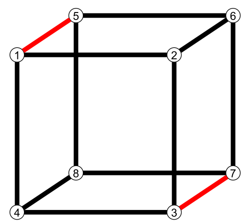
Figure 2.66: A cube with two red edges.



(f) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.67.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_2$.]

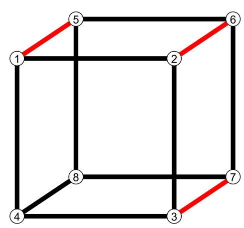
(g) How many distinct ways are there to color two edges of a cube red, leaving the remaining edges black, given that we are allowed to rotate the cube however we like?
[Answer: This will be the number of distinct orbits under our group action. We have already found five distinct orbits, but are not yet sure if there are others. The orbits partition the set of cubes with two red edges. There are twelve choose two or sixty-six cubes in total. As 24+12+12+12+6=66, we know we have found all the orbits. This shows there are five orbits, and thus five distinct ways that we can color two of the edges.]

Figure 2.67: A cube with two red edges.



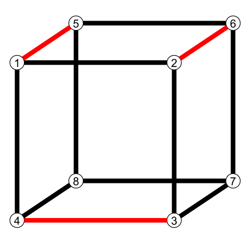
(h) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.68.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong Z_2$.]

Figure 2.68: A cube with three red edges.



(i) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.69.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.69: A cube with three red edges.



(j) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.70.
[Answer: $|orb_G(x)| = 4$, $|stab_G(x)| = 6$,$stab_G(x) \cong D_3$.]

(k) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.71.
[Answer: $|orb_G(x)| = 4$, $|stab_G(x)| = 6$,$stab_G(x) \cong D_3$.]

(l) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.72.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

(m) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.73.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.70: A cube with three red edges.

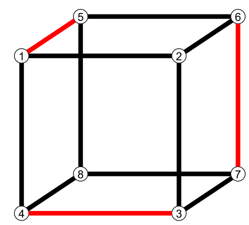


Figure 2.71: A cube with three red edges.

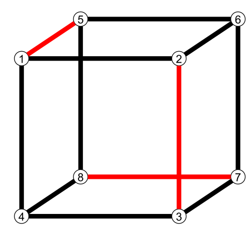


Figure 2.72: A cube with three red edges.

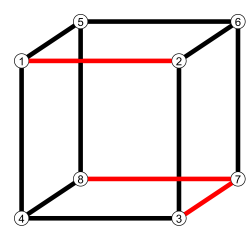
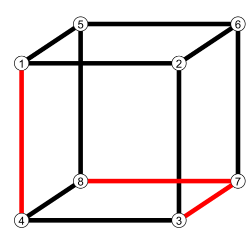


Figure 2.73: A cube with three red edges.



(n) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.74.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

(o) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.75.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

(p) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.76.

Figure 2.74: A cube with three red edges.

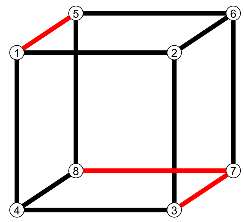
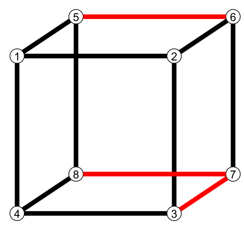
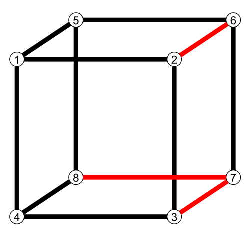


Figure 2.75: A cube with three red edges.



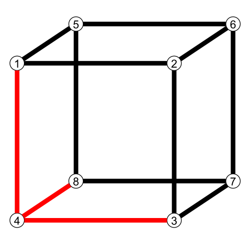
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.76: A cube with three red edges.



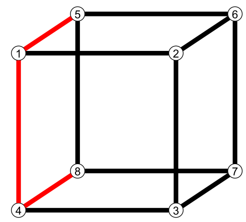
(q) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.77.
[Answer: $|orb_G(x)| = 8$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

Figure 2.77: A cube with three red edges.



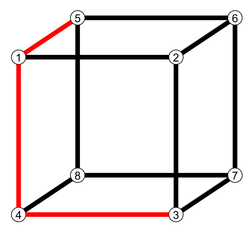
(r) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.78.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.78: A cube with three red edges.



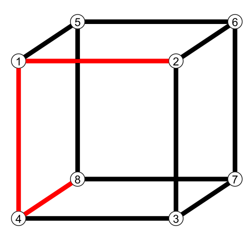
(s) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.79.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.79: A cube with three red edges.



(t) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the cube shown in figure 2.80.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

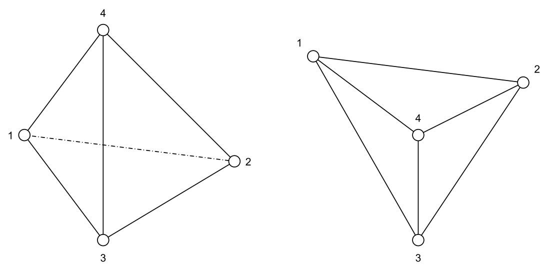
Figure 2.80: A cube with three red edges.



(u) How many distinct orbits do cubes with three red and nine black edges fall into under this action?
[Answer: Let's count the sizes of our orbits so far to see if any are missing. We have found that $12+24+4+4+(24\times 5)+8+24+12+12 = 220$ cubes fall into the orbits belonging to three red edges above. As 220 is the same as twelve choose three, we have all cubes accounted for. Thus there are exactly thirteen orbits. This means that there are exactly thirteen distinct ways to color three edges of a cube.]

2.3.2 The Tetrahedron

Figure 2.81: A white cornered tetrahedron viewed from the side and from above.



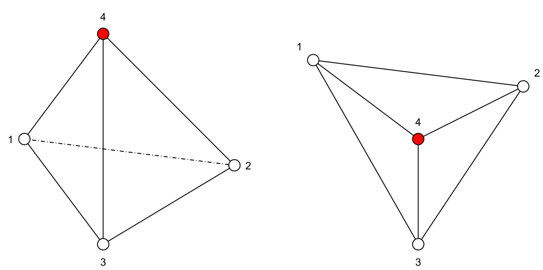
For the following questions let G be the group of rotational symmetries of the tetrahedron. This is the group $G = A_4$ of rotations in 3-space that leave the sides, corners and edges facing the same directions.

1. For the following questions consider the action of G on the set X of all tetrahedra with white and red colored corners.

   (a) Find $|X|$.
   [Answer: We wish to count such colorings for an in place tetrahedron. We get 2^4 as there are four possible corners we must color.]

   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.81.
   [Answer: $|orb_G(x)| = 1$, $|stab_G(x)| = 12$,$stab_G(x) \cong A_4$.]

Figure 2.82: A tetrahedron with one red corner.

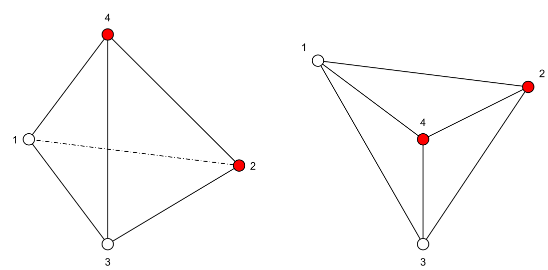


   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.82.
   [Answer: $|orb_G(x)| = 4$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

   (d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.83.
   [Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

   (e) Suppose we are given two terahedron each with exactly two red corners. Is there always a rotation in G mapping the first to the second?
   [Answer: This is the same as asking if there is only one orbit. There are four choose two ways two

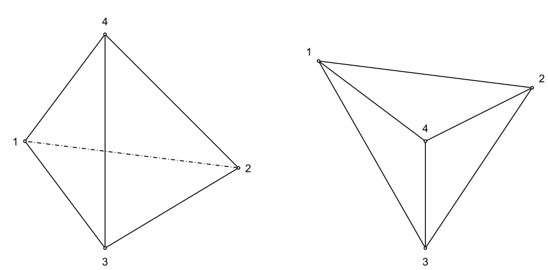
Figure 2.83: A tetrahedron with two red corners.



color two red corners. This means there are six possibilities. As the size of the orbit we found was six, there is only one orbit and the answer is yes.]

(f) How many distinct red and white colorings are there of the corners of a tetrahedron if we are allowed to rotate as we like?
[Answer: The three red and one white case is the same as the one red and three white case due to the bijection that switches the colors red and white. Similarly the four red case is the same as the zero red case. Thus we already have information on all possible numbers of red corners. The orbits we know of have sizes 1,4,6,4 and 1 which add up to all 16 elements of X. Thus we get exactly five orbits, one for each possible number of red corners and our final answer is five.]

Figure 2.84: Side and top view of a black edge tetrahedron.



2. For the following questions consider the action of G on the set X of all tetrahedron with black and purple colored edges.

   (a) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.84.
   [Answer: $|orb_G(x)| = 1$, $|stab_G(x)| = 12$,$stab_G(x) \cong A_4$.]

   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.85.
   [Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.86.
   [Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

   (d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.87.
   [Answer: $|orb_G(x)| = 3$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_2$.]

Figure 2.85: A black and purple edge colored tetrahedron.

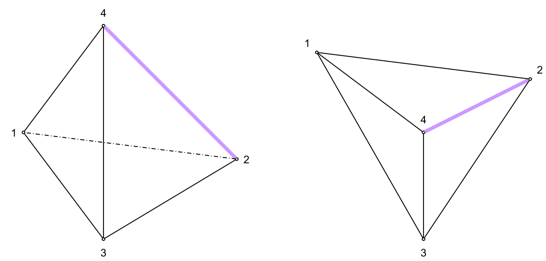


Figure 2.86: A black and purple edge colored tetrahedron.

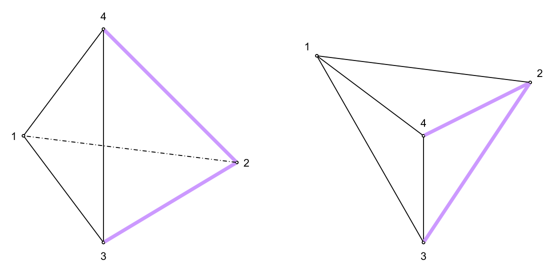
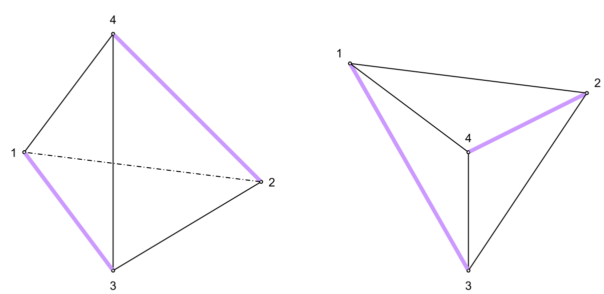


Figure 2.87: A black and purple edge colored tetrahedron..



(e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.88.
[Answer: $|orb_G(x)| = 4$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

(f) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.89.
[Answer: $|orb_G(x)| = 4$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

(g) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.90.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(h) Does every tetrahedron with three black and three purple edges fall into one of the orbits shown from figure 2.88 through figure 2.90?
[Answer: We have six choose three or twenty ways to color a tetrahedron with three black and

Figure 2.88: A three black and three purple edge colored tetrahedron with a corner adjacent to three black edges.

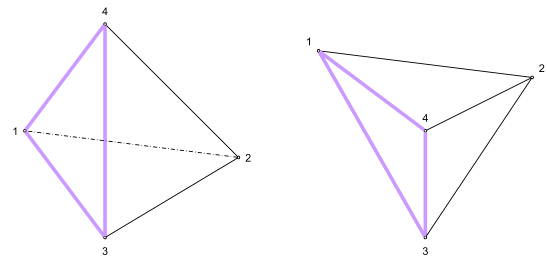


Figure 2.89: A three black and three purple edge colored tetrahedron with a corner adjacent to three purple edges.

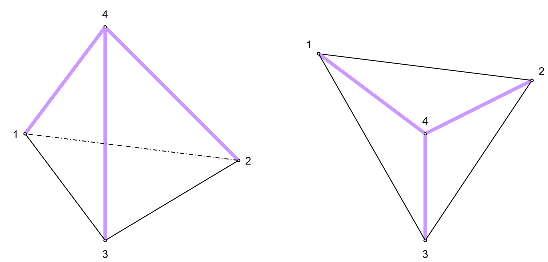
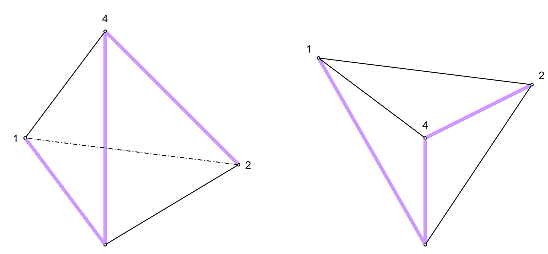


Figure 2.90: A three black and three purple edge colored tetrahedron.

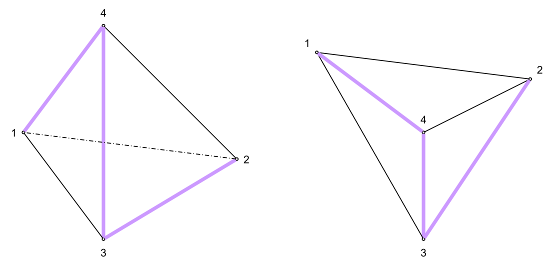


three purple edges. Our orbits are disjoint and have sizes four, four and six respectively. Thus we are missing six tetrahedron that are not yet accounted for and the answer is no.]

(i) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.91.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(j) How many distinct black and purple edge colored tetrahedron are there?
[Answer: We have seen there is one no-purple edged tetrahedron, one one-purple edged tetrahedron, two two-purple edged tetrahedron, and four distinct three-purple edged tetrahedron. For tetrahedron with five and six purple edges, we can just use the data we collected for five black and

Figure 2.91: A three black and three purple edge colored tetrahedron.



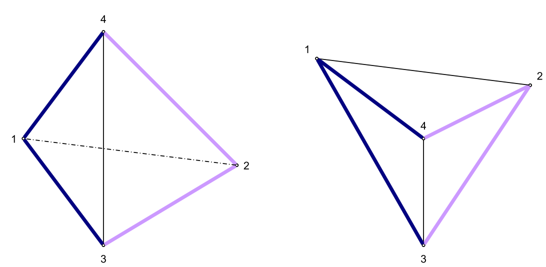
six black edges to give us our final count over all orbits. We get 1+1+2+4+2+1+1 total orbits and thus there are twelve possible distinct edge colorings up to rotation of the tetrahedron.]

3. Consider the collection X of all tetrahedra with two black, two purple and two blue edges. As rotation does not change the number of edges of each color, we can also consider the action of G, the group of rotational symmetries, acting upon this set.

   (a) How many different black, purple and blue edge colorings are there for a tetrahedron if we require there be the same number of each color? In other words, what is the size of X?
   [Answer: We can choose 2 to be black in $C(6,2)$ ways, which leaves $C(4,2)$ ways left to choose two of the remaining edges to be blue. The purple ones are then determined by the others. We therefore get $C(6,2) \times C(4,2) = 15 \times 6 = 90$ different colorings.]

   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.92.
   [Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.92: A tetrahedra with two black, two blue and two purple colored edges.



   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.93.
   [Answer: $|orb_G(x)| = 3$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_2$.]

   (d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.94.
   [Answer: $|orb_G(x)| = 3$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_2$.]

   (e) Are the tetrahedra in figures 2.93 and 2.94 in the same orbit?
   [Answer: No. One way to see this is to picture yourself traveling along the outside of the surface

Figure 2.93: A tetrahedra with two black, two blue and two purple colored edges.

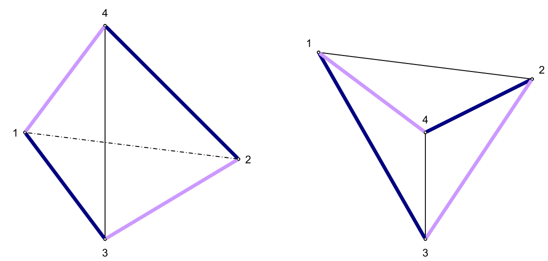
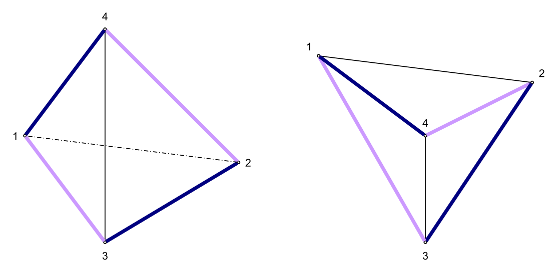


Figure 2.94: A tetrahedra with two black, two blue and two purple colored edges.

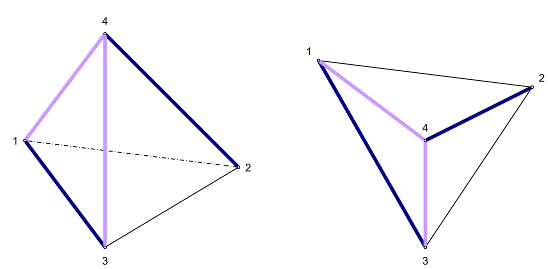


from the midpoint to either end of either of the black edges. In figure 2.93 you will always come to blue edge to your right and purple edge to your left. In figure 2.94 this is reversed.]

(f) Are either of the tetrahedra in figures 2.93 and 2.94 in the orbit of the tetrahedra in figure 2.92?
[Answer: No. If either was, then it would have the same orbit as the tetrahedron in figure 2.92.]

(g) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.95.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

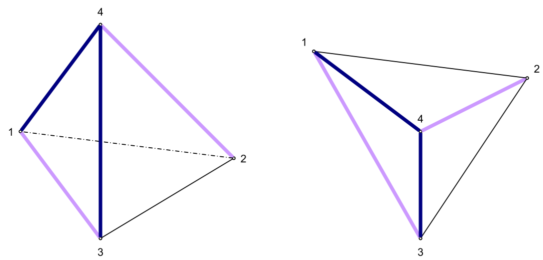
Figure 2.95: A tetrahedra with two black, two blue and two purple colored edges.



(h) Is this tetrahedron in the orbits of the tetrahedra in figures 2.92, 2.93, and 2.94?
[Answer: No. In each of those the two black edges are not adjacent. Rotation cannot change that.]

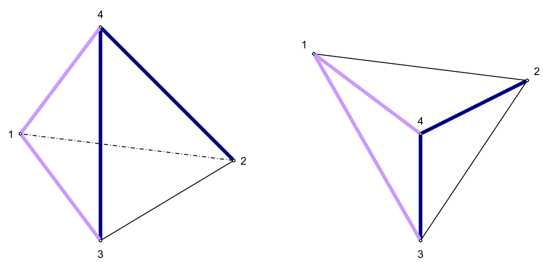
(i) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.96.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.96: A tetrahedra with two black, two blue and two purple colored edges.



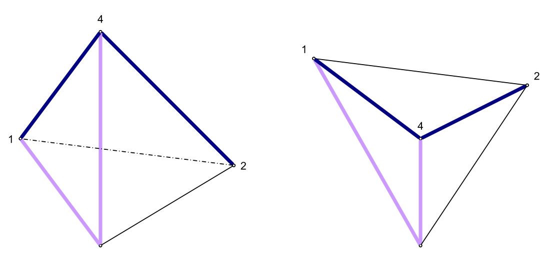
(j) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.97.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.97: A tetrahedra with two black, two blue and two purple colored edges.



(k) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.98.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.98: A tetrahedra with two black, two blue and two purple colored edges.

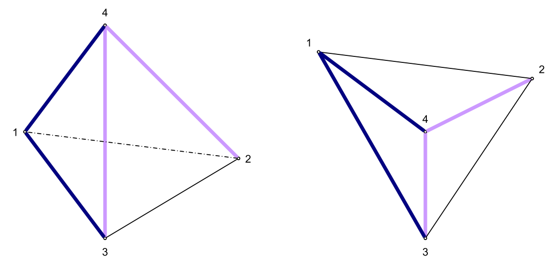


(l) Are the tetrahedra depicted in figures 2.97 and 2.98 in the same orbit?
[Answer: No. Notice that both have a single corner adjacent to one edge of each color. Imagine

traveling from this corner along the black edge in both tetrahedra. In one you will find a the other black edge by turning right after the next corner. In the other you must turn left. Regardless of how you rotate either tetrahedron, this will not change. Thus these taterahedra can not be the same up to rotation.]

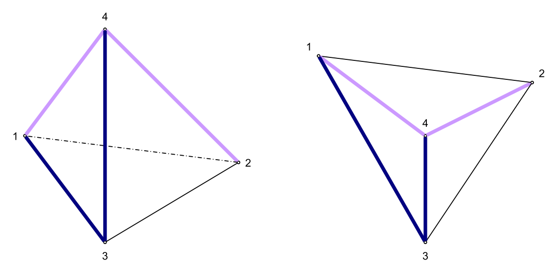
(m) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.99.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.99: A tetrahedra with two black, two blue and two purple colored edges.



(n) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.100.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.100: A tetrahedra with two black, two blue and two purple colored edges.



(o) Are the tetrahedra depicted in figures 2.99 and 2.100 in the same orbit?
[Answer: No. Again, both have a single corner adjacent to one edge of each color. If you were traveling along the outside of both tetrahedra from this corner along the black edge you will find something different once you hit the next corner. In one the other black edge is to the left and in the other, it is to the right. Thus one can not simply be a rotation of the other.]

(p) Are any of the tetrahedra in figures 2.92 through 2.100 in the same orbit?
[Answer: No. Three have black edges which are not adjacent, and we have shown those to be distinct. As for the other six, the one in figure 2.95 is the only of those with non adjacent blue edges and the one in figure 2.96 is the only one with non adjacent purple edges. For the remaining four, two have a blue edge adjacent to both black edges and two have a purple edge adjacent to both black edges. As rotating will fix this property, neither from either pair can be in the same

orbit as something from the other pair. Since we already showed each pair is from a distinct orbit, we now know all orbits are disjoint.]

(q) Is every tetrahedra with two black, two purple, and two blue edges in the orbit of one of the tetrahedra depicted in figures 2.92 through 2.100?
[Answer: Yes. There are 90 total tetrahedra in this set. We have found nine disjoint orbits. Two contain 3 tetrahedra and the other seven contain 12 tetrahedra. Since $2 \times 3 + 7 \times 12 = 90$ we know these orbits must exhaust all possibilities.]

2.3.3 The Octahedron

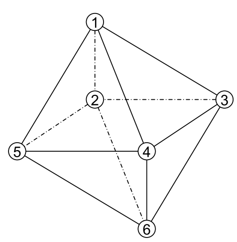
For the following questions let G be the group of rotational symmetries of the octahedron. This is the group $G = S_4$ of rotations in 3-space that leave the sides, corners and edges facing the same directions.

1. For the following questions consider the action of G on the set X of all octahedra with white and orange colored corners.

(a) Find $|X|$.
[Answer: There are six choices, each with two possibilities, so there are $2^6 = 64$ total octahedra.]

(b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.101.
[Answer: $|orb_G(x)| = 1$, $|stab_G(x)| = 24$,$stab_G(x) \cong S_4$.]

Figure 2.101: An octahedron with all white corners.



(c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.102.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 4$,$stab_G(x) \cong \mathbb{Z}_4$.]

Figure 2.102: An octahedron with one orange corner.

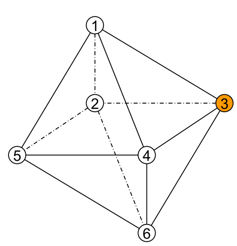
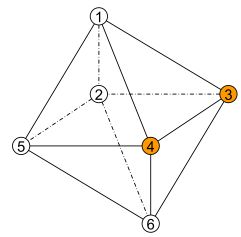


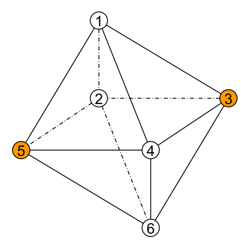
Figure 2.103: An octahedron with two orange corners, adjacent to each other.



(d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.103.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.103.
[Answer: $|orb_G(x)| = 3$, $|stab_G(x)| = 8$,$stab_G(x) \cong D_4$.]

Figure 2.104: An octahedron with two orange corners, not adjacent to each other.



(f) How many octahedra have two orange and four white corners?
[Answer: There are a total of $C(6, 2) = 15$ such octahedra.]

(g) Can every octahedron with two orange and four white corners be rotated to look like either the one in figure 2.103 or figure 2.104?
[Answer: Yes. There are fifteen such octahedra, and the orbits of the two shown are disjoint and contain a total of $12 + 3 = 15$ octahedra.]

(h) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.105.
[Answer: $|orb_G(x)| = 8$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

(i) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.106.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(j) How many octahedra have three orange and three white corners?
[Answer: There are a total of $C(6, 3) = 20$ such octahedra.]

(k) Can every octahedron with three orange and three white corners be rotated to look like either the one in figure 2.105 or figure 2.106?
[Answer: Yes. There are twenty such octahedra, and the orbits of the two shown are disjoint and contain a total of $12 + 8 = 20$ octahedra.]

Figure 2.105: An octahedron with three orange corners, all adjacent to each other.

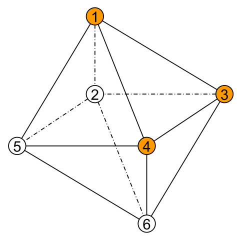
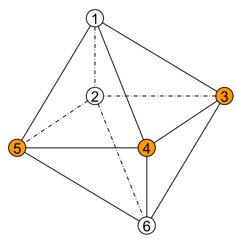


Figure 2.106: An octahedron with three orange corners in a path.



(l) How many distinct octahedra, up to rotations, have four orange and two white corners?
[Answer: This is the same as the number with two orange and four white corners. We already computed that to be two distinct possibilities, with all fifteen such tetrahedra falling into their two orbits.]

(m) How many different ways, up to rotations, can we color the corners of an octahedron in white and orange?
[Answer: There is one way to color it all white and one way to color it all orange. There is one way to do a seven white, one orange split, and one way to do a seven orange, one white split. We have shown there are two ways to color it with two orange, or four orange corners. There are two ways to color it with three white and three orange corners. Thus we have a total of 1+1+2+2+2+1+1 possibilities, giving us ten distinct orange and white cornered octahedra.]

2. For the following questions consider the action of G on the set X of all octahedra with two white, two orange, and two red colored corners.

(a) Find $|X|$.
[Answer: There are $C(6,2)$ ways to pick two corners to be red, and then $C(4,2)$ ways to pick two of the remaining corners to be orange. This gives us a total of $C(6,2)C(4,2) = 15 \times 6 = 90$ octahedra.]

(b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.107.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_4$.]

(c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is any of the octahedra shown in figure 2.108.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.107: An octahedron with opposite corners the same color.

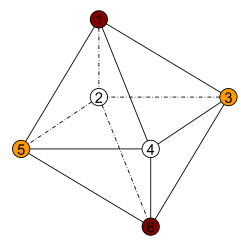
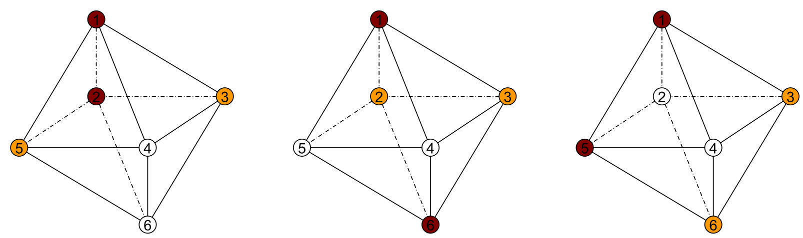


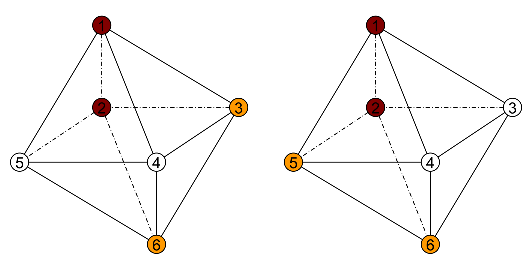
Figure 2.108: Three octahedra each with only one color having opposite corners.



(d) Are any of the octahedra in figure 2.108 in the same orbit?
[Answer: No. Rotating an octahedron doesn't change whether it has opposite corners of a particular color.]

(e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is either of the octahedra shown in figure 2.109.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.109: Two octahedra with no color having opposite corners.



(f) Are the two octahedra in figure 2.109 in the same orbit?
[Answer: No. Think of the opposite corners that are colored red and yellow. There is only one such pair in each case, and the corners correspond to the ones numbered one and six. If we were to try to rotate one onto the other, it would need to rotate along that one-six axis, as those corners are distinguishable. If we rotated the one on the left into the one on the right, we would need to send the red corner labeled two to itself. This leaves us with only the identity, which we can see

does not map the left onto the right.]

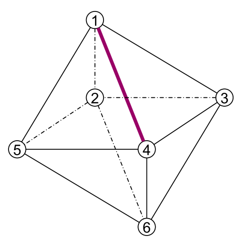
(g) How many distinct colorings of the octahedra have exactly two white, two orange, and two red corners?
[Answer: We have found six octahedra with disjoint orbits. Adding up the sizes of these orbits we get 6+12+12+12+24+24=90. As there are ninety tetrahedra in X in total, every one must fall into one of these six orbits. Thus there are six distinct colorings.]

3. For the following questions consider the action of G on the set X of all octahedra with black and purple colored edges.

(a) Find $|X|$.
[Answer: There are twelve edges each with two possibilities, so there are $2^12 = 4096$ such octahedra.]

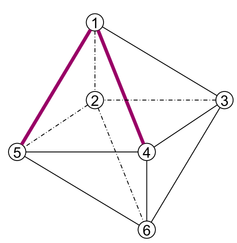
(b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.110.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.110: An octahedron with one purple edge.



(c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.111.
[Answer: $|orb_G(x)| = 24$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

Figure 2.111: An octahedron with two adjacent purple edges.



(d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.112.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is either of the octahedra shown in figure 2.113.
[Answer: $|orb_G(x)| = 12$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.112: An octahedra with two adjacent purple edges.

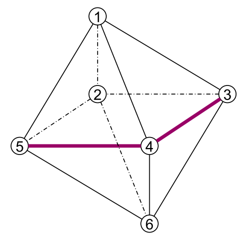
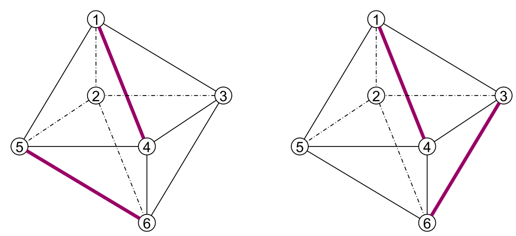


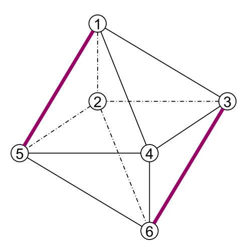
Figure 2.113: Two octahedra each with two purple edges.



(f) Are the two octahedra shown in figure 2.113 in the same orbit?
[Answer: No. Notice that each has a single black edge adjacent to both of the edges that are colored purple. In one picture the purple edges are the ones you would get to from taking left turns from the midpoint of this distinguished black edge. No rotation can change this. In the other picture the purple edges are the ones attainable from right turns instead.]

(g) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the octahedron shown in figure 2.114.
[Answer: $|orb_G(x)| = 6$, $|stab_G(x)| = 4$,$stab_G(x) \cong D_2$.]

Figure 2.114: An octahedron with two purple edges.



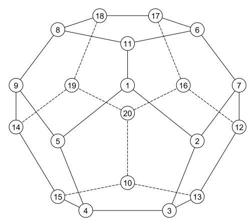
(h) How many ways are there to color the edges of an octahedron with two purple edges and ten black edges? Do not take rotation into account.
[Answer: There are twelve edges so there are $C(12, 2) = 66$ ways.]

(i) Is every octahedra rotationally equivalent to one of the octahedra shown in figures 2.111, 2.112, 2.113, or 2.114?

[Answer: The orbits of these four octahedra have sizes 24, 12, 12, 12, and 6. Each of these are disjoint, and together they contain 66 octahedra. As there are only 66 octahedra with two purple edges, every one must fall into one of these orbits.]

2.3.4 The Dodecahedron

Figure 2.115: A dodecahedron with all white corners.

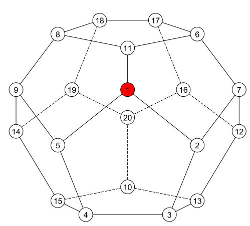


For the following questions let G be the group of rotational symmetries of the dodecahedron. This is the group $G = A_5$ of rotations in 3-space that leave the sides, corners and edges facing the same directions.

1. For the following questions consider the action of G on the set X of all dodecahedra with white and red corners.

   (a) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the dodecahedron shown in figure 2.115.
   [Answer: $|orb_G(x)| = 1$, $|stab_G(x)| = 60$,$stab_G(x) \cong A_5$.]

Figure 2.116: A dodecahedron with one red and nineteen white corners.



   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the dodecahedron shown in figure 2.116.
   [Answer: $|orb_G(x)| = 20$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.117.
   [Answer: $|orb_G(x)| = 30$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

   (d) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.118.
   [Answer: $|orb_G(x)| = 60$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

   (e) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.119.
   [Answer: $|orb_G(x)| = 30$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

   (f) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.120.
   [Answer: $|orb_G(x)| = 30$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

Figure 2.117: A dodecahedron with two red and eighteen white corners.

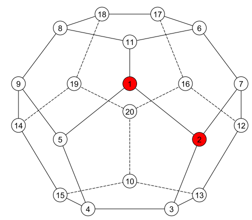


Figure 2.118: A dodecahedron with two red and eighteen white corners.

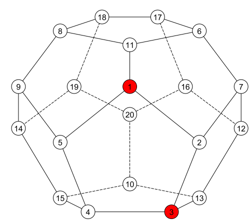


Figure 2.119: A dodecahedron with two red and eighteen white corners.

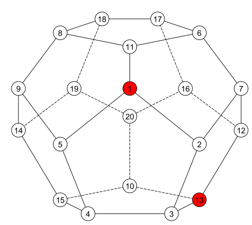
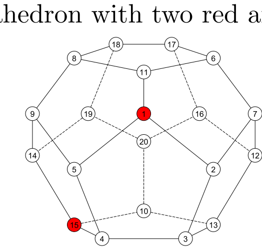


Figure 2.120: A dodecahedron with two red and eighteen white corners.



(g) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.121.
[Answer: $|orb_G(x)| = 30$, $|stab_G(x)| = 2$,$stab_G(x) \cong \mathbb{Z}_2$.]

(h) Find $|orb_G(x)|$, and $|stab_G(x)|$ where x is the tetrahedron shown in figure 2.122.
[Answer: $|orb_G(x)| = 10$, $|stab_G(x)| = 6$,$stab_G(x) \cong D_3$.]

(i) Can any dodecahedra with two red and eighteen white corners be rotated to look like one appearing in figure through 2.117 figure 2.122?

Figure 2.121: A dodecahedron with two red and eighteen white corners.

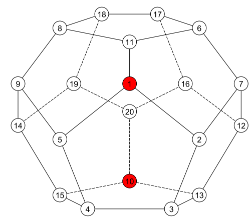
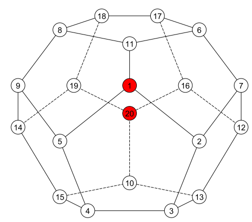


Figure 2.122: A dodecahedron with two red and eighteen white corners.



[Answer: Yes. There are twenty choose two or 190 elements in the set of dodecahedra with two red and eighteen white corners. Our orbits are all disjoint and between them, they contain 30+60+30+30+30+10=190 elements. Thus every possibility is accounted for and the answer is yes.]

2. For the following questions, consider the set of all corner colored dodecahedra with exactly three red corners and seventeen white corners under the group action of the group of rotational symmetries of the dodecahedron.

   (a) Find $|orb_G(x)|$, and $|stab_G(x)|$ for each of the eight dodecahedra in figure 2.123.
   [Answer: The answer is the same in all these cases. We have $|orb_G(x)| = 60$, $|stab_G(x)| = 1$,$stab_G(x) \cong \{e\}$.]

   (b) Find $|orb_G(x)|$, and $|stab_G(x)|$ for the dodecahedron in figure 2.124.
   [Answer: $|orb_G(x)| = 20$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

   (c) Find $|orb_G(x)|$, and $|stab_G(x)|$ for the dodecahedron in figure 2.125.
   [Answer: $|orb_G(x)| = 20$, $|stab_G(x)| = 3$,$stab_G(x) \cong \mathbb{Z}_3$.]

2.3.5 Additional Comments

The case of coloring the faces of the cube is equivalent to that of the conrers of the octahedron. The edge case for the octahedron is the same as the edge case of the cube, and the face case is the same as the corners of the cube. The case for the faces of the tetradehron is the same as that of the corners of the tetrahedron. Therefore, we leave all of these out intentionally. These choices were made because the diagrams for coloring the faces of a platonic solid are less clear than those for the edges and corners.

We included no problems for the icosahedron and a very limited number of problems involving the dodecahedron. As these are dual, so we only need to the edge case for one and the corner case for each to

Figure 2.123: Eight dodecahedra each with three red and eighteen white corners.

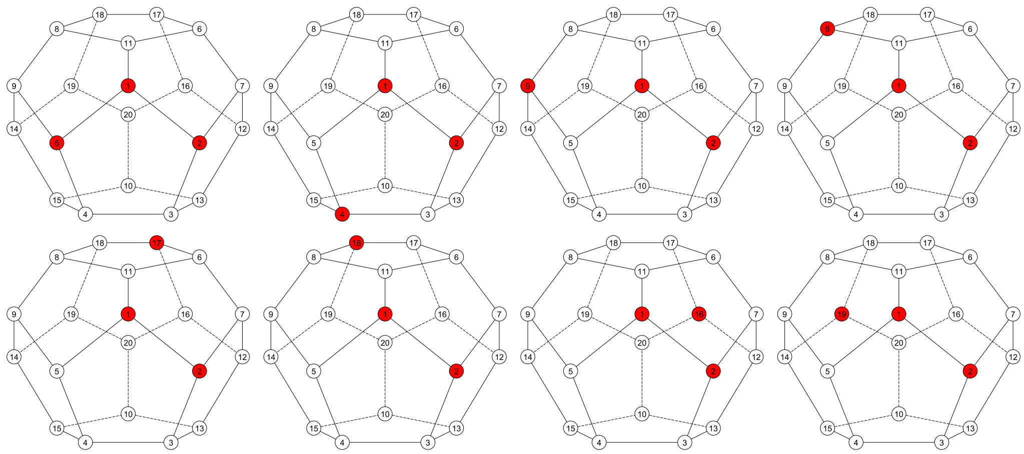


Figure 2.124: A dodecahedron with three red and eighteen white corners.

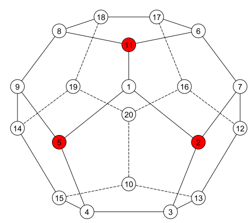
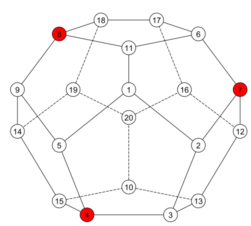


Figure 2.125: A dodecahedron with three red and eighteen white corners.



get full information about the edge colorings, corner colorings and face colorings for both. However due to the large numbers of edges, these calculations do get involved and the Cauchy-Frobenius Theorem is right armound the corner to help us with those.

In most of the problems here, we stuck to exactly two colors for simplicity. This is not needed. Everything can be done with more colors and answers do provide more information as the number of colors increases, until it matches the total number of parts of our solid that we are coloring. Past that it becomes just an issue of picking which colors we are using from a larger set. For example, if we know the arguments for edge colored tetrahedra in six colors, then it is easy to compute information for the seven color case, as we can only use six colors at a time.

2.4 Orbit Stabilizer Questions on Graphs

2.4.1 Graphs and Labeled Graphs on Three Vertices

Let X be the set of all graphs on the set of three labeled vertices and let $G = S_3$ act on this set by permuting the vertices. For example, in a graph containing the edge $\{1,3\}$ the element $(2,3) \in G$ would send that edge to the edge $\{2,3\}$.

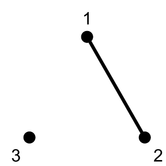
1. For the following questions let x be the graph with no edges.

   (a) If x is the graph with no edges, find $stab(x)$.
   [Answer: All of S_3.]

   (b) Use $|stab(x)|$ to find $|orb(x)|$.
   [Answer: $|stab(x)| = 6, |orb(x)| = \frac{6}{6} = 1$.]

2. For the following questions let x be the graph with the edge $\{1,2\}$ shown in figure 2.126.

Figure 2.126: A graph with one edge.

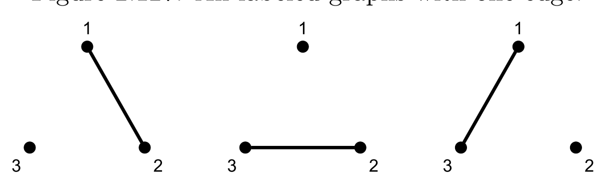


   (a) Find $stab(x)$.
   [Answer: $\{e, (1,2)\}$.]

   (b) Use $|stab(x)|$ to find $|orb(x)|$.
   [Answer: $|stab(x)| = 6, |orb(x)| = \frac{6}{2} = 3$.]

   (c) Find $orb(x)$.
   [Answer: The set of the three graphs shown in figure 2.127.]

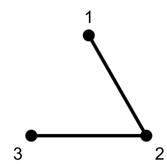
Figure 2.127: All labeled graphs with one edge.



3. For the following questions let x be the graph with the edges $\{1,2\}$ and $\{2,3\}$ shown in figure 2.128.

   (a) Find $stab(x)$.
   [Answer: $\{e, (1,3)\}$.]

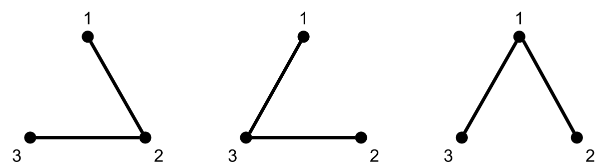
Figure 2.128: A graph with two edges.



(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 6, |orb(x)| = \frac{6}{2} = 3$.]

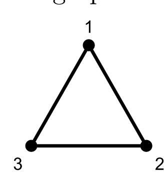
(c) Find $orb(x)$.
[Answer: The set of the three graphs shown in figure 2.129.]

Figure 2.129: All labeled graphs with two edges.



4. For the following questions let x be the graph with the edges $\{1,2\}, \{2,3\}$ and $\{1,3\}$ shown in figure 2.130.

Figure 2.130: A graph with three edges.



(a) Find $stab(x)$.
[Answer: S_3.]

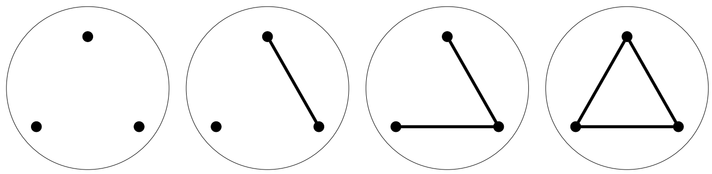
(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 6, |orb(x)| = \frac{6}{6} = 1$.]

(c) Find $orb(x)$.
[Answer: Just the set containing x.]

5. How many graphs are there on three vertices up to isomorphism?
[Answer: We can take one labeled graph in each orbit to see there are four different unlabeled graphs.]

6. Draw all possible unlabeled graphs on three vertices.
[Answer: The set of the graphs shown in figure 2.131.]

Figure 2.131: Unlabeled graphs on three vertices.



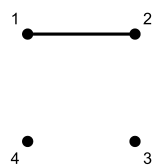
2.4.2 Graphs and Labeled Graphs on Four Vertices

Let X be the set of all graphs on the set of four labeled vertices and let $G = S_4$ act on this set by permuting the vertices.

1. For the following questions let x be the graph with no edges.
    (a) If x is the graph with no edges, find $stab(x)$.
    [Answer: All of S_4.]
    (b) Use $|stab(x)|$ to find $|orb(x)|$.
    [Answer: $|stab(x)| = 24, |orb(x)| = \frac{24}{24} = 1$.]

2. For the following questions let x be the graph with the edge $\{1, 2\}$ shown in figure 2.132.

Figure 2.132: A graph with one edge.



    (a) Find $stab(x)$.
    [Answer: $\{e, (1, 2), (3, 4), (1, 2)(3, 4)\}$.]
    (b) Use $|stab(x)|$ to find $|orb(x)|$.
    [Answer: $|stab(x)| = 4, |orb(x)| = \frac{24}{4} = 6$.]
    (c) Find $orb(x)$.
    [Answer: The set of graphs shown in figure 2.133.]

Figure 2.133: All six labeled graphs with one edge.

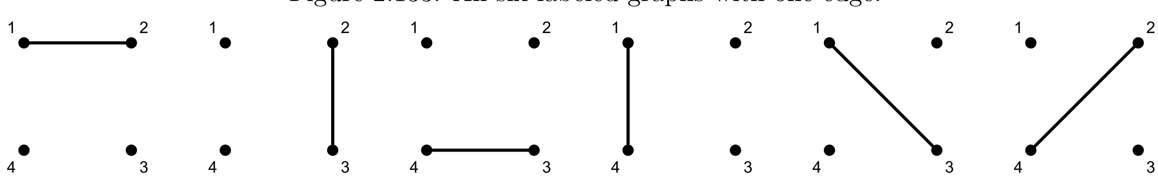
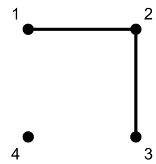


Figure 2.134: A path with two edges.



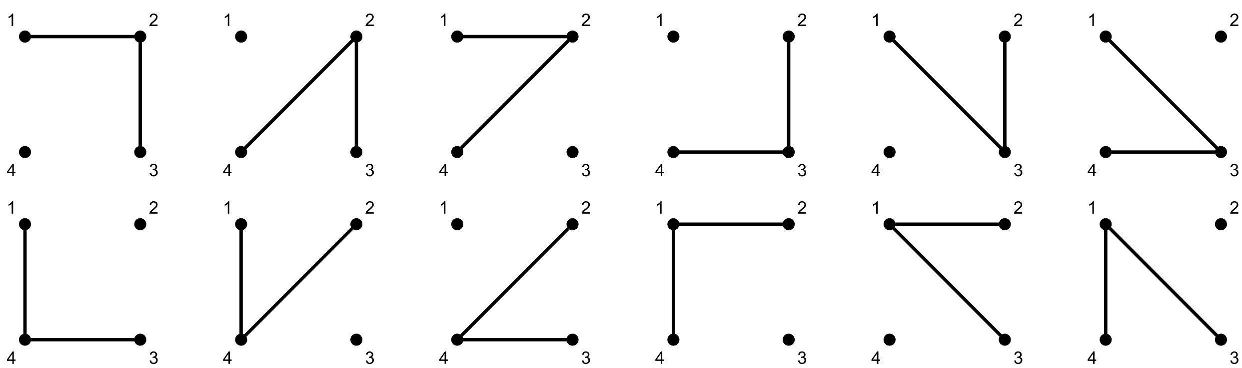
3. For the following questions let x be the graph with edges $\{1, 2\}$ and $\{2, 3\}$ as shown in figure 2.134.

   (a) Find $stab(x)$.
   [Answer: $\{e, (1, 2), (3, 4), (1, 2)(3, 4)\}$.]

   (b) Use $|stab(x)|$ to find $|orb(x)|$.
   [Answer: $|stab(x)| = 4, |orb(x)| = \frac{24}{4} = 6$.]

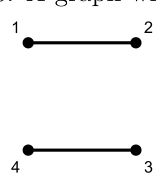
   (c) Find $orb(x)$.
   [Answer: The set of graphs shown in figure 2.135.]

Figure 2.135: Twelve labeled graphs with two edges.



4. For the following questions let x be the graph with the edge $\{1, 2\}$ and $\{3, 4\}$ shown in figure 2.136.

Figure 2.136: A graph with two edges.

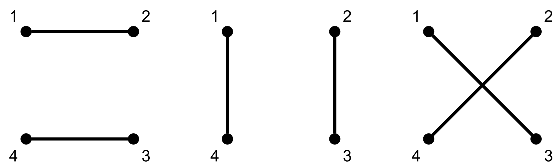


   (a) Find $stab(x)$.
   [Answer: $\{e, (1, 2), (3, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 3, 2, 4), (1, 4, 2, 3)\}$.]

(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 8, |orb(x)| = \frac{24}{8} = 3.$]

(c) Find $orb(x)$.
[Answer: The set of graphs shown in figure 2.137.]

Figure 2.137: Three labeled graphs with two edges.



5. The following questions involve using previous results to find the number of graphs up to isomorphism with two edges on four vertices.

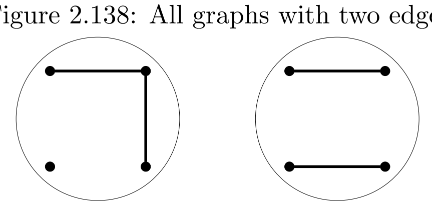
(a) How many labeled graphs are there on four labeled vertices with exactly two edges?
[Answer: There are six possible edges in total, and we need to choose two, so we get $C(6, 2) = 15$ total possible graphs.]

(b) Since elements of G all preserve the number of edges of any given graph, we can consider the action of S_4 on the set of all labeled graphs with two edges on four vertices. How many orbits does this set decompose into?
[Answer: There are twelve graphs in the orbit of the graph shown in 2.132 and three in the orbit of the graph shown in 2.136. As there are only fifteen graphs with two edges in total, this partition contains exactly two disjoint subsets.]

(c) How many unlabeled graphs with two edges are there up to isomorphism?
[Answer: All graphs in each subset of our partition are isomorphic. Thus there are only two distinct graphs up to isomorphism.]

(d) Draw all possible graphs on four vertices with two edges up to isomorphism.
[Answer: See figure 2.138.]

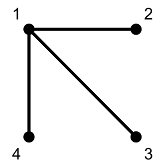
Figure 2.138: All graphs with two edges.



6. For the following questions let x be the graph with edges $\{1, 2\}, \{1, 3\}$ and $\{1, 4\}$ as shown in figure 2.139.

(a) Find $stab(x)$.
[Answer: $\{e, (2, 3, 4), (2, 4, 3), (2, 3), (3, 4), (2, 4)\}$.]

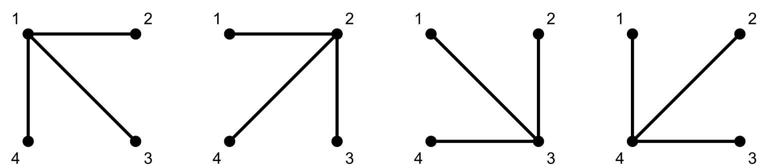
Figure 2.139: A graph with three edges.



(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 6, |orb(x)| = \frac{24}{6} = 4$.]

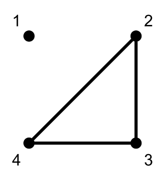
(c) Find $orb(x)$.
[Answer: The set of graphs shown in figure 2.140.]

Figure 2.140: Four labeled graphs with three edges.



7. For the following questions let x be the graph with edges $\{2, 3\}, \{2, 4\}$ and $\{3, 4\}$ as shown in figure 2.151.

Figure 2.141: A graph with three edges.



(a) Find $stab(x)$.
[Answer: $\{e, (2, 3, 4), (2, 4, 3), (2, 3), (3, 4), (2, 4)\}$.]

(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 6, |orb(x)| = \frac{24}{6} = 4$.]

(c) Find $orb(x)$.
[Answer: The set of graphs shown in figure 2.152.]

8. For the following questions let x be the graph with edges $\{1, 2\}, \{2, 3\}$ and $\{3, 4\}$ as shown in figure 2.143.

(a) Find $stab(x)$.
[Answer: $\{e, (1, 4)(2, 3)\}$.]

Figure 2.142: Four labeled graphs with three edges.

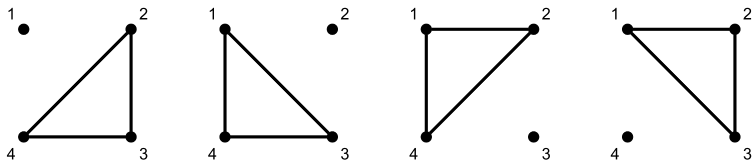
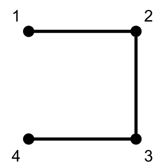


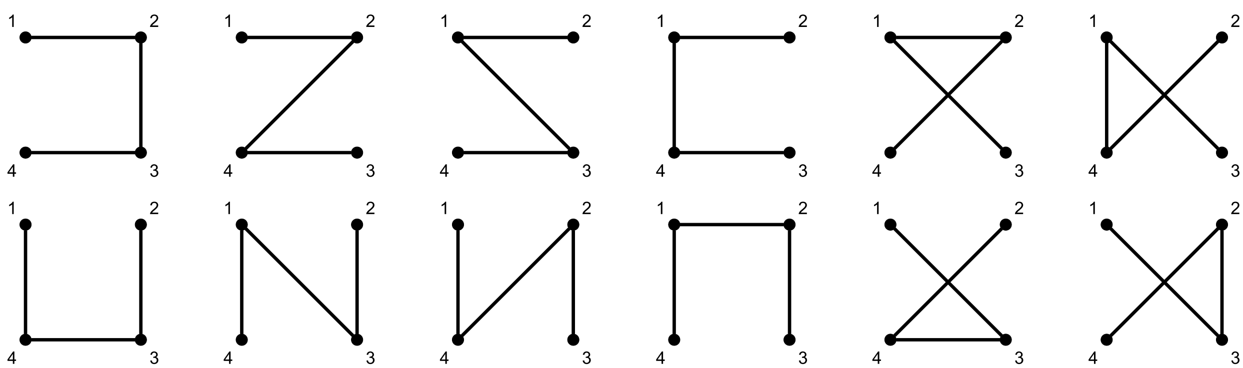
Figure 2.143: A graph with three edges.



(b) Use $|stab(x)|$ to find $|orb(x)|$.
[Answer: $|stab(x)| = 2, |orb(x)| = \frac{24}{2} = 12$.]

(c) Find $orb(x)$.
[Answer: The set of graphs shown in figure 2.144.]

Figure 2.144: Twelve labeled graphs with three edges.



9. The following questions involve using previous results to find the number of graphs up to isomorphism with three edges on four vertices.

(a) How many labeled graphs are there on four labeled vertices with exactly three edges?
[Answer: There are six possible edges in total, and we need to choose three, so we get $C(6, 3) = 20$ total possible graphs.]

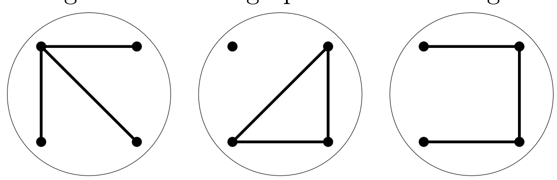
(b) Since elements of G all preserve the number of edges of any given graph, we can consider the action of S_4 on the set of all labeled graphs with two edges on four vertices. How many orbits

does this set decompose into?
[Answer: There are four graphs in the orbit of the graph shown in 2.139, four graphs in the orbit of the graph shown in 2.151, and twelve in the orbit of the graph shown in 2.143. As there are only twenty graphs with two edges in total, this partition contains exactly three disjoint subsets.]

(c) How many unlabeled graphs with three edges are there up to isomorphism?
[Answer: All graphs in each subset of our partition are isomorphic. Thus there are only three distinct graphs up to isomorphism.]

(d) Draw all possible graphs on four vertices with three edges up to isomorphism.
[Answer: See figure 2.145.]

Figure 2.145: All graphs with three edges.



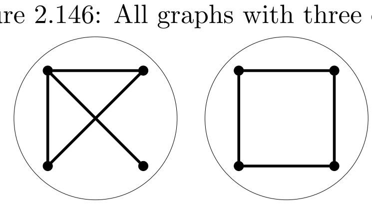
10. The following questions involve using previous results to find the number of graphs up to isomorphism with four edges on four vertices.

    (a) How many labeled graphs are there on four labeled vertices with exactly four edges?
    [Answer: There are six possible edges in total, and we need to choose four, so we get $C(6,4) = 15$ total possible graphs.]

    (b) How many unlabeled graphs with four edges are there up to isomorphism?
    [Answer: Here we can use the fact that there is a bijection between graphs with two edges and graphs with all but two of their edges. The compliment map places an edge between vertices if, and only if, there was no edge before. Not only does this map form a bijection between labeled graphs with two edges to labeled graphs with four edges, it does the same for unlabeled graphs as well. Since there are only two unlabeled graphs on two edges, there must be only two unlabeled graphs with four edges.]

    (c) Draw all possible graphs on four vertices with four edges up to isomorphism.
    [Answer: See figure 2.146.]

Figure 2.146: All graphs with three edges.



11. The following questions involve using previous results to find the total number of graphs up to isomorphism on four vertices.

(a) How many labeled graphs are there on four labeled vertices with exactly five edges?
[Answer: There are six possible edges in total, and we need to choose four, so we get $C(6,5) = 6$ total possible graphs.]

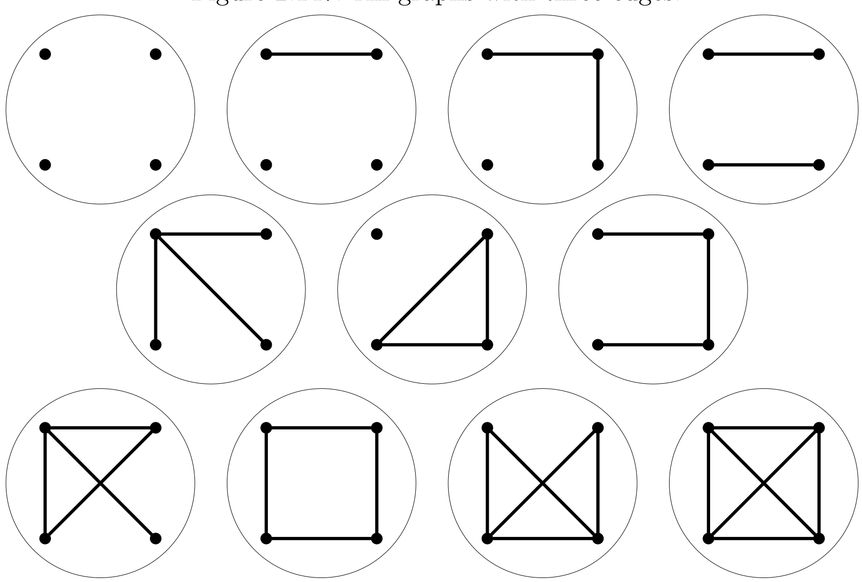
(b) How many labeled and unlabeled graphs with five edges are there up to isomorphism?
[Answer: The compliment map is a bijection taking graphs and labeled graphs with one edge to graphs and labeled graphs with five edges. There are therefore six labeled graphs with five edges, and one unlabeled graph with five edges.]

(c) How many graphs are there on four labeled vertices with exactly six edges?
[Answer: Labeled or unlabeled all edges must be included. There can be only one of each.]

(d) Up to isomophism, how many unlabeled graphs are there on four vertices?
[Answer: We now know from our previous results that there is only one graph each in the cases of zero, one, five and six edges. We also know there are two graphs in the two edge and four edge cases, and three graphs in the three edge case. This gives us a total of $1+1+2+3+2+1+1 = 11$.]

(e) Draw all possible graphs on four vertices up to isomorphism.
[Answer: See figure 2.147.]

Figure 2.147: All graphs with three edges.



2.5 The Proof of the Orbit Stabilizer Theorem

1. In this first set of exercises prove the Orbit-Stabilizer Theorem in full generality. The theorem states that for any group action of G on the set X, and for any element x in X, we have

$$|G| = |orb(x)| \cdot |stab(x)|.$$

Throughout these exercises we let $H = stab(x)$ and label the distinct cosets $H, a_2H, a_3H, \cdots, a_nH$. We call the set of distinct cosets C and define a map $\phi : C \to orb(x)$ by setting $\phi(bH) = b \cdot x$ where the product on the right hand side represents the action of b on x. We assume knowledge of the fact that the distinct cosets of a subgroup of a group form a partition of that group, though we include other necessary basic coset lemmas as exercises. We use the term coset to mean left coset. Therefore the cosets of a subgroup H are the sets of the form $aH = \{ah \in G : h \in H\}$.

(a) Prove that if J is any subgroup of G and $J = gJ$ for some $g \in G$ then $g \in J$.
[Answer: Since the two sets J and gJ are equal, we know that e is in gJ and thus $gj = e$ for some $j \in J$. This means g is j^{-1} and since subgroups contain the inverses of all their elements, we see g must be in J.]

(b) Prove that ϕ is a well defined map. This means that it takes all elements of each distinct coset to the same place. If it didn't do this, then it wouldn't be a map from C at all, as the image would depend on the way the coset was represented and not the coset itself.
[Answer: Suppose bH and cH are the same coset. This means that the sets bH and cH contain the same elements even though b and c may not be the same element of G. We wish to show that $\phi(bH)$ equals $\phi(cH)$. As $\phi(bH) = b \cdot x$ and $\phi(cH) = c \cdot x$ we need to show that $b \cdot x = c \cdot x$. Since $bH = cH$ we know that as sets $b^{-1}(bH) = b^{-1}(cH)$ which means that $H = b^{-1}cH$, which can only happen if $b^{-1}c$ is in H (by our last exercise.) We only need to use this fact with the properties of group actions to see that $c\dot{x} = (ec)\dot{x} = (bb^{-1}c)\dot{x} = b((b^{-1}c) \cdot x) = b \cdot x$.]

(c) Prove that ϕ is injective. This means showing that if $\phi(bH)$ equals $\phi(cH)$ then bH equals cH. This does not mean b equals c, of course, but only that the two sets bH and cH contain the same elements.
[Answer: Assume $\phi(bH) = \phi(cH)$, which means $b \cdot x = c \cdot x$. This implies $b^{-1} \cdot (b \cdot x) = b^{-1} \cdot (c \cdot x)$ which means $(b^{-1}b) \cdot x = (b^{-1}c) \cdot x$. Since this tells us that $x = e \cdot x = (b^1 b) \cdot x = (b^{-1}c) \cdot x$ we know that $b^{-1}c$ is in the stabilizer. Thus $b^{-1}c \in H$.
Take any g in bH. Then $g = bh_1$ for some h_1 in H. As $h_1 = b^{-1}cc^{-1}bh_1$ we can write h_1 as $b^{-1}ch_2$ for h_2 in H. Thus $g = bh_1 = bb^{-1}ch_2 = ch_2$ which shows that g is in cH. This argument shows that $bH \subseteq cH$ and switching b and c gives us the proof for the other direction as well.]

(d) Prove that ϕ is surjective. This means that if y is in $orb(x)$ then there is some bH so that $\phi(bH) = y$.
[Answer: Assume $y \in orb(x)$. Then $y = g \cdot x$ for some $g \in G$. Then by the definition of ϕ we known $\phi(gH) = g \cdot X = y$.]

(e) Prove that any two cosets of a subgroup in a finite group have the same size.
[Answer: Let J be any subgroup of G. We will show all cosets of J have size J. Let kJ be any coset of J. Then consider the map $f : kJ \to J$ defined by $f(g) = k^{-1}g$. The map is injective because if $f(b) = f(c)$, then $k^{-1}b = k^{-1}c$, and $b = c$ by the left cancellation. The map is surjective because if j is any element of J then the element kj is in kJ and is sent by f to j. Thus we have a bijection from kJ to J, which shows that kJ and J have the same size.]

(f) Prove that $|G| = |orb(x)| \cdot |stab(x)|$.
[Answer: We know that there is a bijection between C, the set of all cosets, and the set $orb(x)$ and thus $|C| = |orb(x)|$. If we can show $|C| = |G|/|stab(x)|$ then we are done. Since the cosets of H partition G, we must have $|G| = |H| + |a_1H| + |a_2H| + \cdots + |a_mH|$. Since each coset has the same size, this sum equals $|C| \times |H|$. Since $|G| = |C| \times |H|$ we know $|C| = |C||H|/|H| = |G|/|H|$ which completes the proof.]

2. In this set of exercises we examine the bijection ϕ between the cosets of the stabilizer and the orbit of some element x, where G is the group D_4 acting on the set of all two-by-two matrices with entries in $\mathbb{Z}_2$. Assume that f reflects a matrix about a vertical axis of symmetry and r rotates a matrix by ninety degrees.

   (a) Let x be the element $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

      i. Find $stab(x)$.
      [Answer: The stabilizer of x is the set $\{e, f\}$.

      ii. List the cosets of the subgroup $stab(x)$ of G.
      [Answer: Set $H = stab(x)$. We then have $H = \{e, f\}$, $rH = \{r, rf\}$, $r^2H = \{r^2, r^2f\}$, and $r^3H = \{r^3, r^3f\}$. Note that $fH = H$, $(rf)H = rH$, $(r^2f)H = r^2H$, and $(r^3f)H = r^2H$. Any list containing exactly one of H and fH, one of rH and $(rf)H$, one of r^2H and $(r^2f)H$, and one of r^3H and $(r^3f)H$ would form a complete list of cosets.]

      iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
      [Answer: We have seen that $|C| = 4$.]

      iv. Find $orb(x)$.
      [Answer: $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

      v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
      [Answer: $\phi(H) = e \cdot x = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\phi(rH) = r \cdot x = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\phi(r^2H) = r^2 \cdot x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\phi(r^3H) = r^3 \cdot x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Note that we also could have taken $f \cdot x$ to find $\phi(H)$ since $f \in H$, but we get the same result. Similarly, we could have taken $(rf) \cdot x$ for $\phi(rH)$, $(r^2f) \cdot x$ for $\phi(r^2H)$, and $(r^3f) \cdot x$ for $\phi(r^3H)$ and the results, of course, would be the same.]

   (b) Let x be the element $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

      i. Find $stab(x)$.
      [Answer: The stabilizer of x is the set $\{e, rf, r^2, r^3f\}$.

      ii. List the cosets of the subgroup $stab(x)$ of G.
      [Answer: Set $H = stab(x)$. We then have $H = (rf)H = r^2H = (r^3f)H = \{e, rf, r^2, r^3f\}$, and $fH = rH = (r^2f)H = r^3H = \{f, r, r^2f, r^3\}$. We can simply write H, rH for a complete list, or use other representatives and coset notation, or list the sets $\{e, rf, r^2, r^3f\}$,$\{f, r, r^2f, r^3\}$ to answer the question.]

      iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
      [Answer: We have seen that $|C| = 2$.]

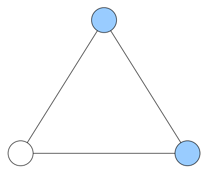
      iv. Find $orb(x)$.
      [Answer: $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

      v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
      [Answer: $\phi(H) = e \cdot x = (rf) \cdot x = r^2 \cdot x = (r^3f) \cdot x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\phi(fH) = f \cdot x = r \cdot x = (r^2f) \cdot x = r^3 \cdot x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.]

3. In this set of exercises we examine the bijection ϕ between the cosets of the stabilizer and the orbit of some element x, where G is the group D_3 acting on the set of all white and blue necklaces with three beads. Assume that f reflects a necklace about a vertical axis of symmetry and r rotates a necklace clockwise by 120 degrees.

   (a) Let x be the necklace shown in figure 2.148.

   Figure 2.148: A white and blue necklace on three vertices.

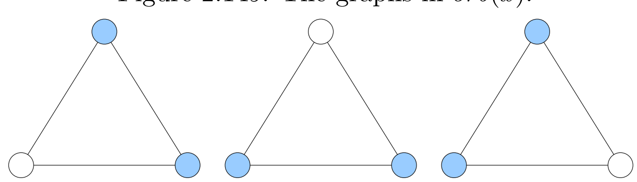
   i. Find $stab(x)$.
   [Answer: The stabilizer of x is the set $\{e, rf\}$.

   ii. List the cosets of the subgroup $stab(x)$ of G.
   [Answer: Set $H = stab(x)$. We then have $rH = \{r, r^2 f\}$ and $r^2H = \{r^2, rf\}$. We can list H, rH, r^2H for our cosets, use different representatives for the three cosets, or list the sets themselves.]

   iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
   [Answer: We have seen that $|C| = 3$.]

   iv. Find $orb(x)$.
   [Answer: The set $orb(x)$ consists of the three necklaces pictured in figure 2.149.

   Figure 2.149: The graphs in $orb(x)$.

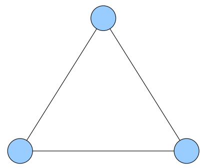
   v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
   [Answer: $\phi(H)$ is the first necklace depicted in figure 2.149, $\phi(rH)$ is the second, and $\phi(r^2H)$ is the third.]

   (b) Let x be the necklace shown in figure 2.150.

   i. Find $stab(x)$.
   [Answer: The stabilizer of x is all of D_3.

   ii. List the cosets of the subgroup $stab(x)$ of G.
   [Answer: We only have one coset $H = stab(x)$. This could be written as eH, rH, r^2H, fH, rfH, or r^2fH, but they are all the same set.]

Figure 2.150: A labeled graph on three vertices.



iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
[Answer: We have seen that $|C| = 1$.]

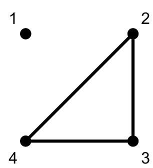
iv. Find $orb(x)$.
[Answer: The set $orb(x)$ consists only of the graph in figure 2.150.

v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
[Answer: $\phi(H)$ goes to the necklace in figure 2.150, the only member in $orb(x)$.]

4. In this set of exercises we examine the bijection ϕ between the cosets of the stabilizer and the orbit of some element x, where G is the group S_4 acting on the set of all labeled graphs on four vertices.

(a) Let x be the element shown in figure 2.151.

Figure 2.151: A labeled graph.



i. Find $stab(x)$.
[Answer: The stabilizer of x is the set $\{e, (2,3), (2,4), (3,4), (2,3,4), (2,4,3)\}$.

ii. List the cosets of the subgroup $stab(x)$ of G.
[Answer: Set $H = stab(x)$. We then have $H = \{e, (2,3), (2,4), (3,4), (2,3,4), (2,4,3)\}$, $(1,2)H = \{(1,2), (1,2,3), (1,2,4), (1,2)(3,4), (1,2,3,4), (1,2,4,3)\}$, $(1,3)H = \{(1,3), (1,3,2), (1,3)(2,4), (1,3,4), (1,3,4,2), (1,2,4,3)\}$, and $(1,4)H = \{(1,4), (1,4)(2,3), (1,4,2), (1,4,3), (1,4,2,3), (1,4,3,2)\}$. We can list $H, (1,2)H$, $(1,3)H$, $(1,4)H$ for our cosets, use different representatives for the four cosets, or list the sets themselves.]

iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
[Answer: We have seen that $|C| = 4$.]

iv. Find $orb(x)$.
[Answer: The set $orb(x)$ consists of the four graphs pictured in figure 2.152.

v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
[Answer: $\phi(H)$ is the first graph depicted in figure 2.152, $\phi((1,2)H)$ is the second, and $\phi((1,3)H)$ and $\phi((1,4)H)$ are the third and fourth.]

(b) Let x be the element shown in figure 2.153.

Figure 2.152: The graphs in $orb(x)$.

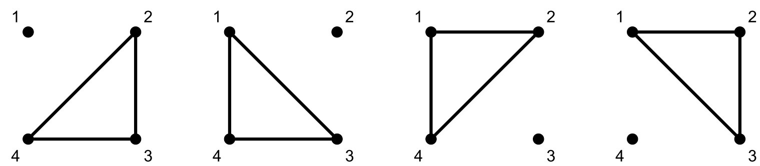
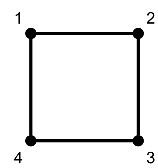


Figure 2.153: A labeled graph.



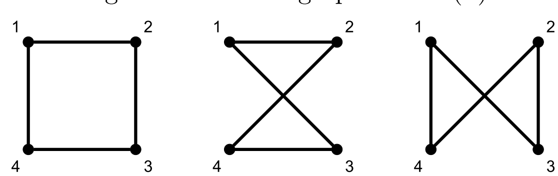
i. Find $stab(x)$.
[Answer: The stabilizer of x is the set $\{e, (1,2,3,4), (1,4,3,2), (1,3)(2,4), (1,3), (2,4), (1,2)(3,4), (1,3)(2,4)\}$.

ii. List the cosets of the subgroup $stab(x)$ of G.
[Answer: Set $H = stab(x)$. We then have $H = \{e, (1,2,3,4), (1,4,3,2), (1,3)(2,4), (1,3), (2,4), (1,2)(3,4), (1,3)(2,4)\}$, $(1,2)H = \{(1,2), (2,3,4), (1,4,3), (1,3,2,4), (1,3,2), (1,2,4), (3,4), (1,3,2,4)\}$, and $(1,4)H = \{(1,4), (1,2,3), (2,4,3), (1,3,4,2), (1,3,4), (1,4,2), (1,2,4,3), (1,3,4,2)\}$. We can list H, $(1,2)H$, $(1,4)H$ for our cosets, use different representatives for these three cosets, or list the sets themselves.]

iii. Find $|C| = |G|/|stab(x)|$ where C is the set of all cosets of the subgroup $stab(x)$ of G.
[Answer: We have seen that $|C| = 3$.]

iv. Find $orb(x)$.
[Answer: The set $orb(x)$ consists of the four graphs pictured in figure 2.154.

Figure 2.154: The graphs in $orb(x)$.



v. Find $\phi(c)$ for every c in the set of all cosets of $stab(x)$.
[Answer: $\phi(H)$ is the first graph depicted in figure 2.154, $\phi((1,2)H)$ is the second, and and $\phi((1,4)H)$ is the third.]

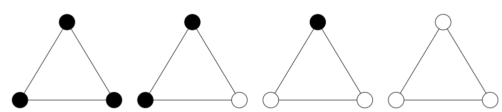
Chapter 3

The Cauchy-Frobenius Lemma

3.1 Necklaces

1. How many necklaces are there, up to rotations and reflections, with three beads of two possible colors?
[Answer: Our group G is $D_3 \equiv S_3$ For any of the three 2-cycles g, $|X^g| = 2^2 = 4$. For either of the two 3-cycles g, $|X^g| = 2^1 = 2$. For the identity $|X^g| = 2^3 = 8$. We then get $\frac{1}{|G|}(3\times4+2\times2+1\times8) = \frac{24}{6} = 4$. Thus there are four distinct necklaces.]

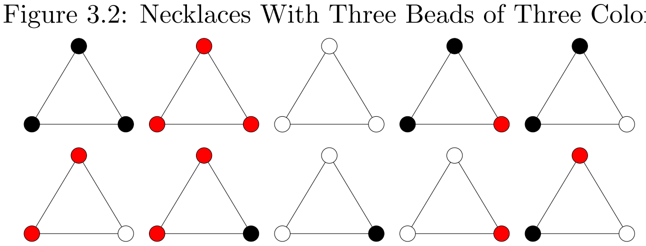
Figure 3.1: Necklaces With Three Beads of Two Colors



2. Draw these necklaces.
[Answer: See figure 3.1.]

3. How many necklaces are there, up to rotations and reflections, with three beads of three possible colors?
[Answer: Our group G is $D_3 \cong S_3$. For any of the three 2-cycles g, $|X^g| = 3^2 = 9$. For either of the two 3-cycles g, $|X^g| = 3^1 = 3$. For the identity $|X^g| = 3^3 = 27$. We then get $\frac{1}{|G|}(3\times9+2\times3+1\times27) = \frac{60}{6} = 10$. Thus there are ten distinct necklaces.]

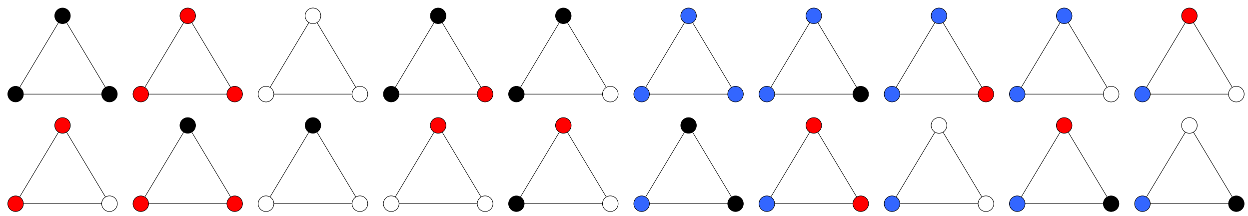
Figure 3.2: Necklaces With Three Beads of Three Colors



4. Draw these necklaces.
[Answer: See figure 3.2.]

5. How many necklaces are there, up to rotations and reflections, with three beads of four possible colors?
[Answer: Our group G is $D_3 \cong S_3$. For any of the three 2-cycles g, $|X^g| = 4^2 = 16$. For either of the two 3-cycles g, $|X^g| = 4^1 = 4$. For the identity $|X^g| = 4^3 = 64$. We then get $\frac{1}{|G|}(3 \times 16 + 2 \times 4 + 1 \times 64) = \frac{120}{6} = 20$. Thus there are ten distinct necklaces.]

Figure 3.3: Necklaces With Three Beads of Four Colors

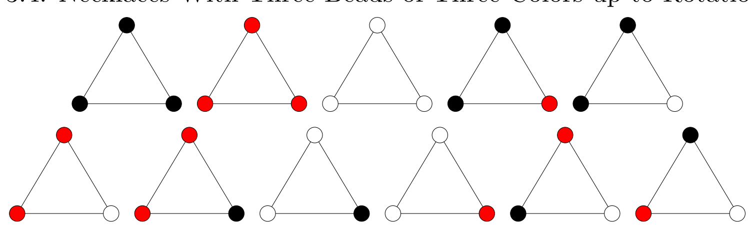


6. Draw these necklaces.
[Answer: See figure 3.3.]

7. How many necklaces are there, up to rotations and reflections, with three beads of n possible colors?
[Answer: Our group G is $D_3 \equiv S_3$. For any of the three 2-cycles g, $|X^g| = n^2$. For either of the two 3-cycles g, $|X^g| = n^1$. For the identity $|X^g| = n^3$. We then get $\frac{1}{|G|}(3n^2 + 2n + n^3) = \frac{1}{6}n(n+1)(n+2)$.]

8. How many necklaces are there, up to only rotations, with three beads of three possible colors?
[Answer: Our group G is $\{e, r, r^2\} \cong \mathbb{Z}_3$. For both r and r^2, $|X^g| = 3^1$. For the identity $|X^g| = 3^3$. We then get $\frac{2 \times 3 + 3^3}{3} = 11$.]

Figure 3.4: Necklaces With Three Beads of Three Colors up to Rotation Alone

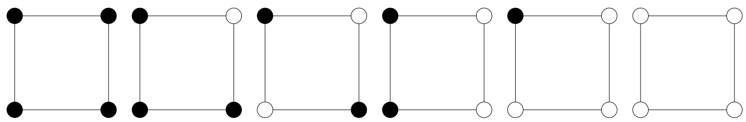


9. Draw these necklaces.
[Answer: See figure 3.4.]

10. How many necklaces are there, up to only rotations, with three beads of n possible colors?
[Answer: Our group G is $\{e, r, r^2\} \cong \mathbb{Z}_3$. For both r and r^2, $|X^g| = n^1$. For the identity $|X^g| = n^3$. We then get $\frac{1}{|G|}(2n + n^3) = \frac{1}{3}n(n^2 + 2)$.]

11. How many necklaces are there, up to rotations and reflections, with four beads of two possible colors?
[Answer: Our group G is D_4. For r and r^3, $|X^g| = 2^1$. For r^2, $|X^g| = 2^2 = 4$. For a reflection about the two axes through two opposite beads, $|X^g| = 2^3 = 8$. For a reflection about the two axes between beads, $|X^g| = 2^2 = 4$. For the identity $|X^g| = 2^4 = 16$. We then get $\frac{1}{|G|}(2\times2+1\times4+2\times8+2\times4+16) = \frac{48}{8} = 6$. Thus there are six distinct necklaces.]

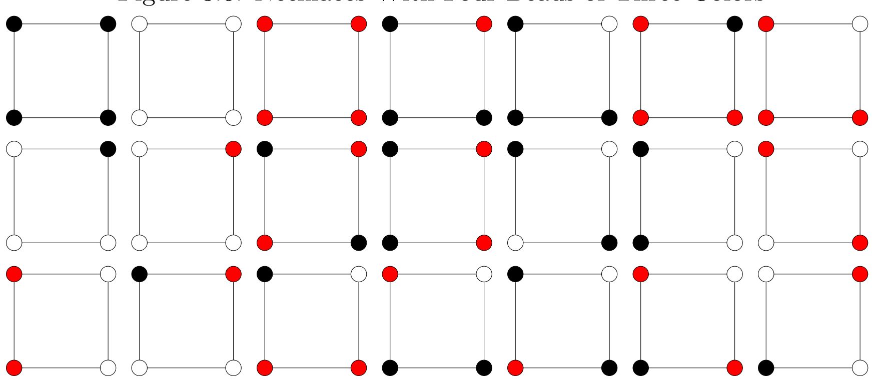
Figure 3.5: Necklaces With Four Beads of Two Colors



12. Draw these necklaces.
[Answer: See figure 3.5.]

13. How many necklaces are there, up to rotations and reflections, with four beads of three possible colors?
[Answer: Our group G is D_4. For r and r^3, $|X^g| = 3^1$. For r^2, $|X^g| = 3^2 = 9$. For a reflection about the two axes through two opposite beads, $|X^g| = 3^3 = 27$ For a reflection about the two axes between beads, $|X^g| = 3^2 = 9$. For the identity $|X^g| = 3^4 = 81$. We then get $\frac{1}{|G|}(2 \times 3 + 1 \times 9 + 2 \times 27 + 2 \times 9 + 81) = \frac{168}{8} = 21$. Thus there are twenty-one distinct necklaces.]

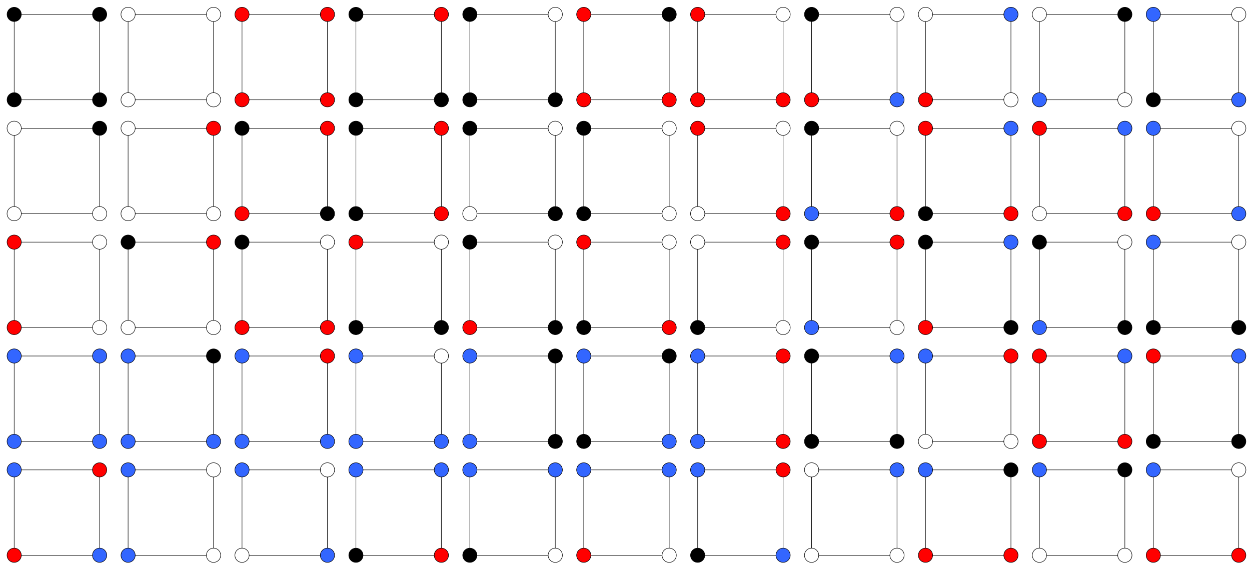
Figure 3.6: Necklaces With Four Beads of Three Colors



14. Draw these necklaces.
[Answer: See figure 3.6.]

15. How many necklaces are there, up to rotations and reflections, with four beads of four possible colors?
[Answer: Our group G is D_4. For r and r^3, $|X^g| = 4^1$. For r^2, $|X^g| = 4^2 = 16$. For a reflection about the two axes through two opposite beads, $|X^g| = 4^3 = 64$ For a reflection about the two axes between beads, $|X^g| = 4^2 = 16$. For the identity $|X^g| = 4^4 = 256$. We then get $\frac{1}{|G|}(2\times4+1\times16+2\times64+2\times16+256) = \frac{440}{8} = 55$. Thus there are fifty-five distinct necklaces.]

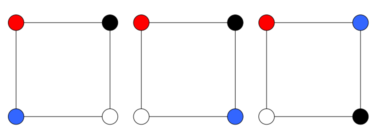
Figure 3.7: Necklaces With Four Beads of Four Colors



16. Draw these necklaces.
[Answer: See figure 3.7.]

17. How many necklaces are there, up to rotations and reflections, with one bead in each of four different colors?
[Answer: Our group G is D_4. Here, no element that moves beads will fix anything, as each bead is distinct. Thus $|X^g| = 0$ for all g except the identity. For the identity $|X^g| = 4! = 24$. We then get $\frac{1}{8}(24) = 3$. Thus there are three distinct necklaces.]

Figure 3.8: Necklaces with One Bead in Each of Four Colors

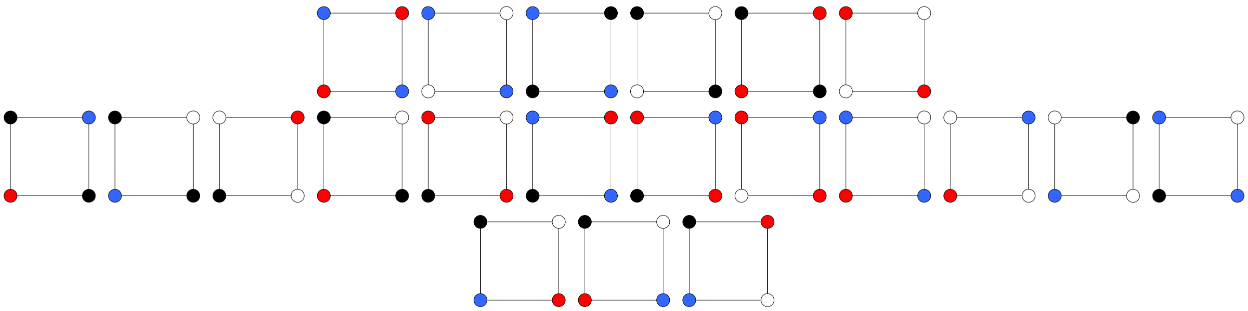


18. Draw these necklaces.
[Answer: See figure 3.8.]

19. How many necklaces with four beads of four possible colors, up to rotations and reflections, have the property that no two beads in a row have the same color?
[Answer: Our group G is D_4. Anything fixed under r, r^3 or a reflection between two beads must have $|X^g| = 0$. For r^2 we get $|X^g| = 4 \times 3 = 12$. For reflections through two beads we get $|X^g| = 4 \times 3 \times 3 = 36$. For the identity we have to carefully count necklaces with no two beads in a row. We

get $|X^g| = 4 \times 3 \times 3 + 4 \times 3 \times 2 \times 2 = 84$. We then get $\frac{1}{8}(12 + 2 \times 36 + 84) = 21$. Thus there are twenty-one such necklaces.]

Figure 3.9: Four Colored Necklaces with No Two Beads in a Row the Same



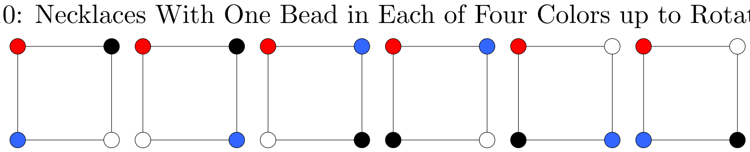
20. Draw these necklaces.
[Answer: See figure 3.9.]

21. How many necklaces are there, up to rotations and reflections, with four beads of n possible colors?
[Answer: Our group G is D_4. For r and r^3, $|X^g| = n^1$. For r^2, $|X^g| = n^2$. For a reflection about the two axes through two opposite beads, $|X^g| = n^3$ For a reflection about the two axes between beads, $|X^g| = n^2$. For the identity $|X^g| = n^4$. We then get $\frac{1}{|G|}(2n+n^2+2n^3+2n^2+n^4) = \frac{1}{8}n(n+1)(n^2+n+2)$.]

22. How many necklaces are there, up to rotations alone, with four beads of n possible colors?
[Answer: Our group G is $\{e, r, r^2, r^3\} = \mathbb{Z}_4$. For r and r^3, $|X^g| = n^1$. For r^2, $|X^g| = n^2$. For the identity $|X^g| = n^4$. We then get $\frac{1}{|G|}(2n + n^2 + n^4) = \frac{1}{4}n(n+1)(n^2 - n + 2)$.]

23. How many necklaces are there, up to rotations alone, with four beads in each of four different colors?
[Answer: Our group G is $\mathbb{Z}_4$ but our set contains only necklaces with all distinct beads. Here, no element that moves beads will fix anything, as each bead is distinct. Thus $|X^g| = 0$ for all g except the identity. For the identity $|X^g| = 4! = 24$. We then get $\frac{1}{4}(24) = 6$. Thus there are six distinct necklaces.]

Figure 3.10: Necklaces With One Bead in Each of Four Colors up to Rotations Alone

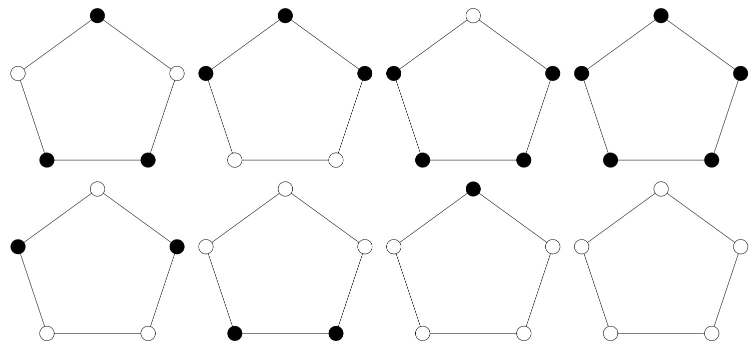


24. Draw these necklaces.
[Answer: See figure 3.10.]

25. How many necklaces are there, up to rotations and reflections, with five beads of two possible colors?
[Answer: Our group G is D_5. For r, r^2, r^3 and r^4, $|X^g| = 2^1$. Each of the five reflections have

$|X^g| = 2^3 = 8$. For the identity $|X^g| = 2^5 = 32$. We then get $\frac{1}{|G|}(4 \times 2 + 5 \times 8 + 32) = \frac{80}{10} = 8$. Thus there are eight distinct necklaces.]

Figure 3.11: Necklaces with Five Beads of Two Colors



26. Draw these necklaces.
[Answer: See figure 3.11.]

27. How many blue and green necklaces in five beads are there, up to rotations and reflections, with more blue beads than green beads?
[Answer: We know switching green and blue beads gives us a bijection that maps no necklace to itself. Therefore exactly half the eight necklaces with two colored beads must have this property, giving us four possibilities. This sort of argument can also be used to show that the number of two colored necklaces with an odd number of beads must always be even.]

28. How many necklaces are there, up to rotations and reflections, with five beads of three possible colors?
[Answer: Our group G is D_5. For r, r^2, r^3 and r^4, $|X^g| = 3^1$. Each of the five reflections have $|X^g| = 3^3 = 27$. For the identity $|X^g| = 3^5 = 243$. We then get $\frac{1}{|G|}(4 \times 3 + 5 \times 27 + 243) = \frac{390}{10} = 39$. Thus there are thirty-nine distinct necklaces.]

29. Draw these necklaces.
[Answer: See figure 3.12.]

30. How many necklaces are there, up to rotations and reflections, with five beads of n possible colors?
[Answer: Our group G is D_5. For r, r^2, r^3 and r^4, $|X^g| = n^1$. Each of the five reflections have $|X^g| = n^3$. For the identity $X^g = n^5$. We then get $\frac{1}{|G|}(4n + 5n^3 + n^5) = \frac{1}{10}n(n^2+1)(n^2+4)$.]

31. How many necklaces are there, up to rotations and reflections, with exactly two green and three blue beads?
[Answer: Our group G is D_5, but our set only includes possibilities with the exact numbers of beads described. For r, r^2, r^3 and r^4, we now get $|X^g| = 0$. Each of the five reflections have $|X^g| = 2$. For the identity $X^g = C(5, 2) = 10$. We then get $\frac{1}{|G|}(5 \times 2 + 10) = \frac{20}{10} = 2$. Thus there are two distinct necklaces.]

32. Draw these necklaces.
[Answer: See figure 3.13.]

Figure 3.12: Necklaces with Five Beads of Three Colors

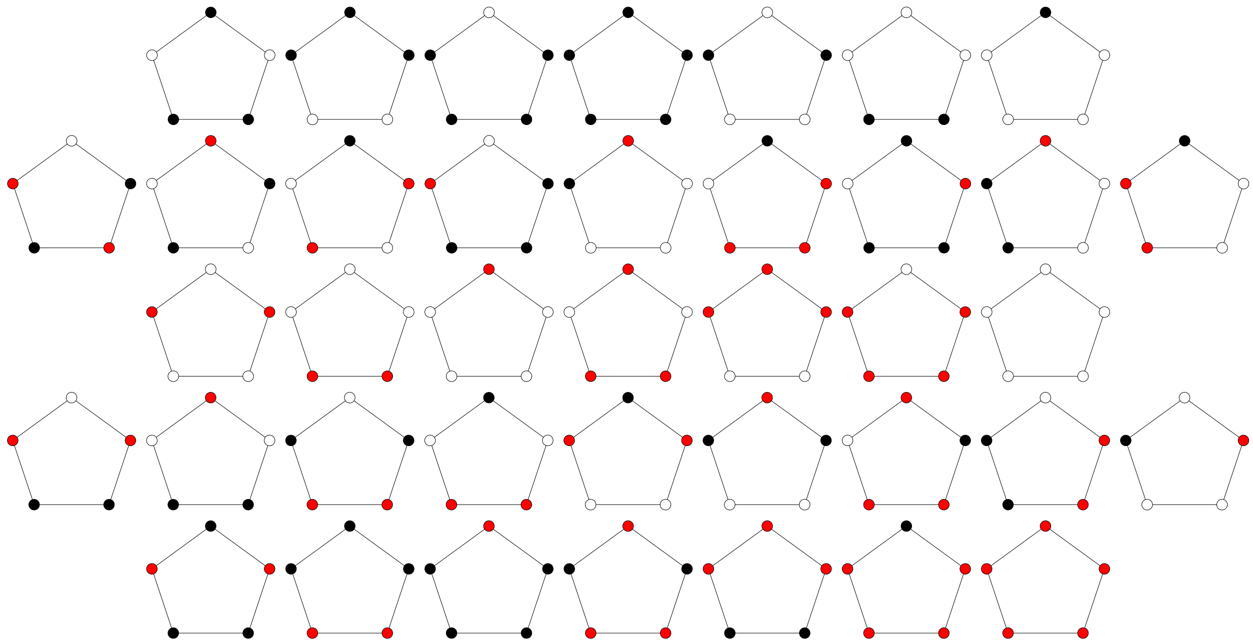
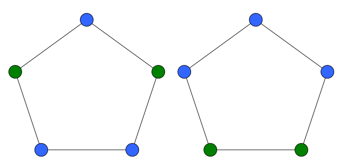
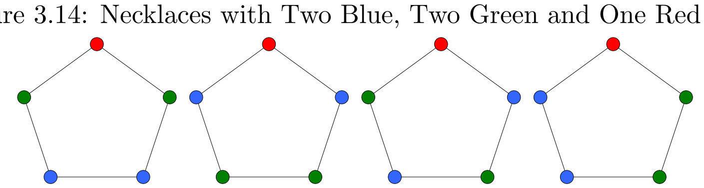


Figure 3.13: Necklaces with Two Green and Three Blue Beads



33. How many necklaces are there, up to rotations and reflections, with exactly two blue, two green and one red bead?
[Answer: Our group G is D_5. For r, r^2, r^3 and r^4, we get $|X^g| = 0$. Each of the five reflections have $|X^g| = 2$. For the identity $|X^g| = C(5,2)C(3,2)C(1,1) = 30$. We then get $\frac{1}{|G|}(5 \times 2 + 30) = \frac{40}{10} = 4$. Thus there are four distinct necklaces.]

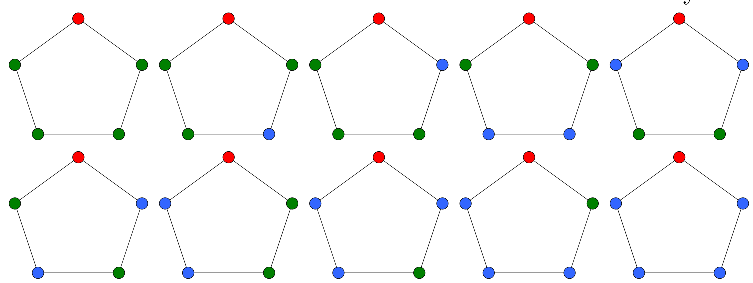
Figure 3.14: Necklaces with Two Blue, Two Green and One Red Bead



34. Draw these necklaces.
[Answer: See figure 3.14.]

35. How many necklaces, up to rotations and reflections, have exactly one red bead, and four others which can be any combination of blue or green?
[Answer: Our group G is D_5. Our set is necklaces with exactly one red bead. For r, r^2, r^3 and r^4, we get $|X^g| = 0$. For each reflection, the axis must go through the red bead in order for it to be fixed. This gives us $|X^g| = 2^2$. Finally for e we have $|X^g| = 5 \times 2^4$ as there are five places for our red bead and the others each have two choices. This gives us $\frac{1}{|G|}(5 \times 4 + 80) = \frac{100}{10} = 10$. Thus there are ten necklaces meeting our criterion.]

Figure 3.15: Necklaces with Five Beads of Three Colors with Exactly One Red Bead



36. Draw these necklaces.
[Answer: See figure 3.15.]

37. How many necklaces are there, up to rotations alone, with five beads of n possible colors?
[Answer: Our group G is $\mathbb{Z}_5$. For r, r^2, r^3 and r^4, $|X^g| = n^1$. For the identity $X^g = n^5$. We then get $\frac{1}{|G|}(4n + n^5) = \frac{1}{5}n(n^2 - 2n + 2)(n^2 + 2n + 2)$.]

38. How many necklaces, up to rotations alone, have exactly two green and three blue beads?
[Answer: Our group G is $\mathbb{Z}_5$. Our set is necklaces with exactly one red bead. For r, r^2, r^3 and r^4, we get $|X^g| = 0$. For e we have $|X^g| = C(5, 2) = 10$ This gives us $\frac{1}{|G|}(5) = \frac{10}{5} = 2$. Thus there are two such necklaces.]

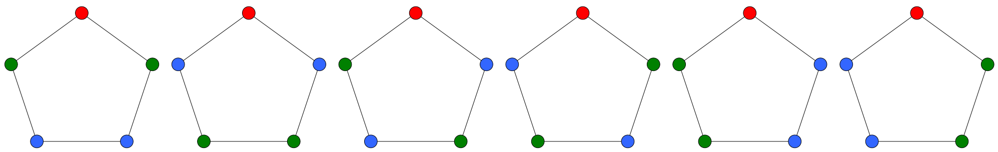
39. Draw these necklaces.
[Answer: These are the same as in the dihedral case. See figure 3.13.]

40. How many necklaces, up to rotations alone, have exactly one red, two green and two blue beads?
[Answer: Our group G is $\mathbb{Z}_5$. Our set is necklaces with exactly one red bead. For r, r^2, r^3 and r^4, we get $|X^g| = 0$. For e we have $|X^g| = C(5, 2)C(3, 2) = 30$ This gives us $\frac{1}{|G|}(5) = \frac{30}{5} = 6$. Thus there are six such necklaces.]

41. Draw these necklaces.
[Answer: See figure 3.16.]

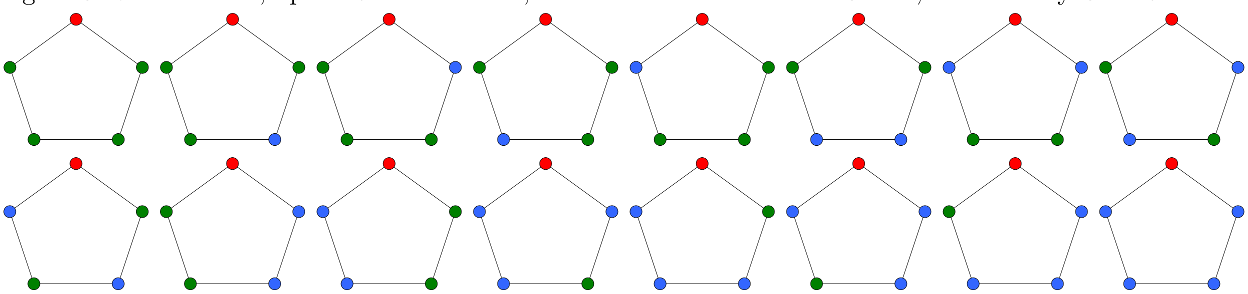
42. How many necklaces, up to rotations alone, have exactly one red bead, and four others which can be any combination of blue or green?

Figure 3.16: Necklaces up to Rotation with Two Blue, Two Green and One Red Bead.



[Answer: Our group G is $\mathbb{Z}_5$. Our set is necklaces with exactly one red bead. For r, r^2, r^3 and r^4, we get $|X^g| = 0$. For e we have $|X^g| = 5 \times 2^4$ as there are five places for our red bead and the others each have two choices. This gives us $\frac{1}{|G|}(80) = \frac{80}{5} = 16$. Thus there are sixteen necklaces meeting our criterion.]

Figure 3.17: Necklaces, up to Rotation Alone, with Five Beads of Three Colors, and Exactly One Red Bead



43. Draw these necklaces.
[Answer: See figure 3.17.]

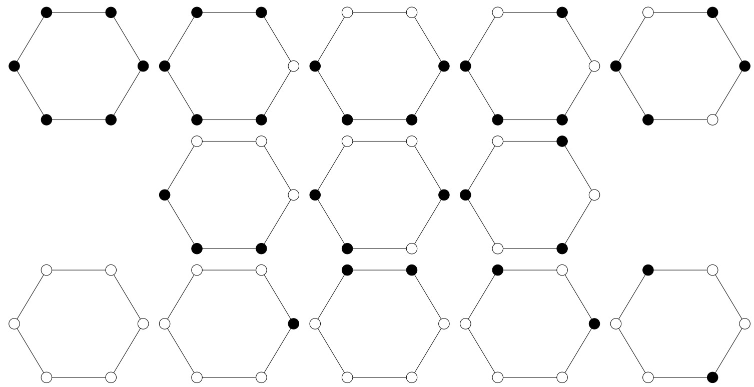
44. How many necklaces are there, up to rotations and reflections, with six beads of two possible colors?
[Answer: Our group G is D_6. For r and r^5, $|X^g| = 2^1$. For r^2 and r^4, $|X^g| = 2^2$. For r^3, $|X^g| = 2^3$. Each of the three reflections between beads have $|X^g| = 2^3 = 8$. Each of the three reflections through opposite beads have $|X^g| = 2^4 = 16$. For the identity $X^g = 2^6 = 64$. We then get $\frac{1}{|G|}(2 \times 2^1 + 2 \times 2^2 + 1 \times 2^3 + 3 \times 2^3 + 3 \times 2^4 + 2^6) = \frac{156}{12} = 13$. Thus there are eight distinct necklaces.]

45. Draw these necklaces.
[Answer: See figure 3.18.]

46. How many necklaces are there, up to rotations and reflections, with six beads of three possible colors?
[Answer: Our group G is D_6. For r and r^5, $|X^g| = 3^1$. For r^2 and r^4, $|X^g| = 3^2$. For r^3, $|X^g| = 3^3$. Each of the three reflections between beads have $|X^g| = 3^3$. Each of the three reflections through opposite beads have $|X^g| = 3^4$. For the identity $X^g = 3^6$. We then get $\frac{1}{|G|}(2 \times 3^1 + 2 \times 3^2 + 1 \times 3^3 + 3 \times 3^3 + 3 \times 3^4 + 3^6) = \frac{1104}{12} = 92$. Thus there are 92 distinct necklaces.]

47. How many necklaces are there, up to rotations and reflections, with six beads of n possible colors?
[Answer: Our group G is D_6. For r and r^5, $|X^g| = n^1$. For r^2 and r^4, $|X^g| = n^2$. For r^3, $|X^g| = n^3$.

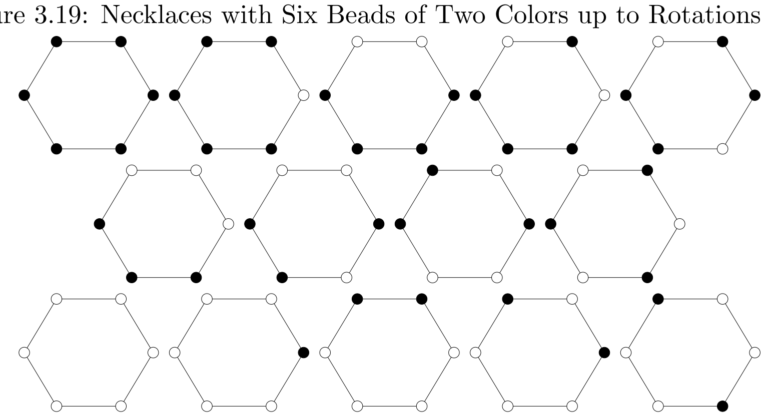
Figure 3.18: Necklaces with Six Beads of Two Colors



Each of the three reflections between beads have $|X^g| = n^3$. Each of the three reflections through opposite beads have $|X^g| = n^4$. For the identity $|X^g| = n^6$. We then get $\frac{1}{|G|}(2n^1 + 2n^2 + n^3 + 3n^3 + 3n^4 + n^6) = \frac{1}{12}n(n+1)(n^4 - n^3 + 4n^2 + 2)$.]

48. How many necklaces are there, up to rotations alone, with six beads of two possible colors?
[Answer: Our group G is $\mathbb{Z}_6$. For r and r^5, $|X^g| = 2^1$. For r^2 and r^4, $|X^g| = 2^2$. For r^3, $|X^g| = 2^3$. For the identity $|X^g| = 2^6$. We then get $\frac{1}{|G|}(2 \times 2^1 + 2 \times 2^2 + 1 \times 2^3 + 2^6) = \frac{84}{6} = 14$.]

Figure 3.19: Necklaces with Six Beads of Two Colors up to Rotations Alone



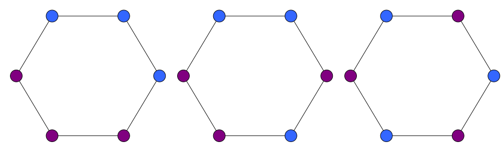
49. Draw these necklaces.
[Answer: See figure 3.19.]

50. How many necklaces are there, up to rotations alone, with six beads of n possible colors?
[Answer: Our group G is $\mathbb{Z}_6$. For r and r^5, $|X^g| = n^1$. For r^2 and r^4, $|X^g| = n^2$. For r^3, $|X^g| = n^3$. For the identity $|X^g| = n^6$. We then get $\frac{1}{|G|}(2n^1 + 2n^2 + n^3 + n^6) = \frac{1}{6}n(n+1)(n^4 - n^3 + n^2 + 2)$.]

51. How many necklaces are there, up to rotations and reflections, with three blue and three purple beads?

[Answer: Our group G is D_6. For r, r^3, and r^5, $|X^g| = 0$. For r^2 and r^4, $|X^g| = 2$. For reflections about a line through two beads we have $|X^g| = 4$, but for other reflections $|X^g| = 0$ For the identity $|X^g| = C(6,3) = 20$. We thus get $\frac{1}{|G|}(2 \times 2 + 3 \times 4 + 20) = \frac{36}{12} = 3$. Thus there are three distinct necklaces.]

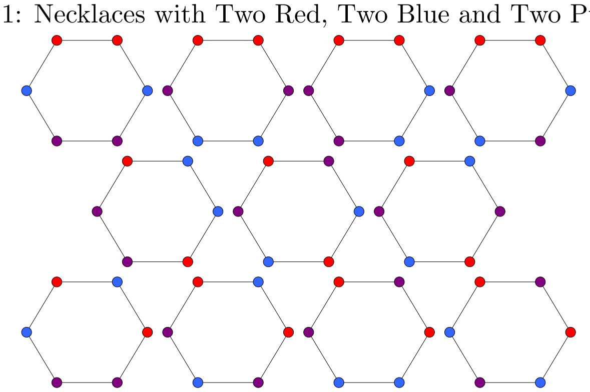
Figure 3.20: Necklaces with Three Blue and Three Purple beads



52. Draw these necklaces.
    [Answer: See figure 3.20.]

53. How many necklaces are there, up to rotations and reflections, with two red, two blue and two purple beads?
    [Answer: Our group G is D_6. For r, r^2, r^4 and r^5, $|X^g| = 0$. For r^3, $|X^g| = 6$. Each of the six reflections we get $|X^g| = 6$ regardless of whether the line of reflection goes through or between the beads. For the identity $|X^g| = C(6,2)C(4,2)C(2,2) = 90$. We thus get $\frac{1}{|G|}(7 \times 6 + 90) = \frac{132}{12} = 11$. Thus there are eleven distinct necklaces.]

Figure 3.21: Necklaces with Two Red, Two Blue and Two Purple beads

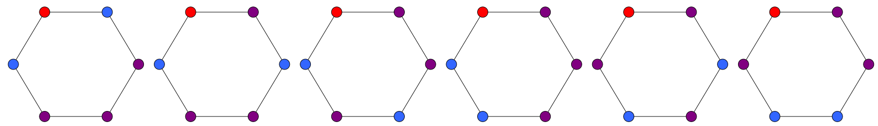


54. Draw these necklaces.
    [Answer: See figure 3.21.]

55. How many necklaces are there up to rotations and reflections, with one red, two blue and three purple beads?
    [Answer: Our group G is D_6. For all rotations, $|X^g| = 0$. For reflections between beads we have $|X^g| = 0$ but for reflections about a line through two beads we get $|X^g| = 4$. For the identity $|X^g| = C(6,3)C(3,2)C(1,1) = 60$. We thus get $\frac{1}{|G|}(3 \times 4 + 60) = \frac{72}{12} = 6$. Thus there are six distinct necklaces.]

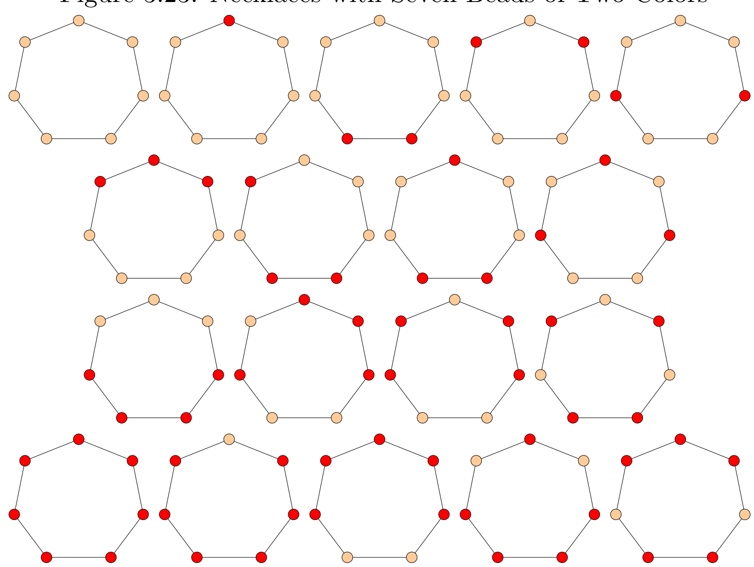
56. Draw these necklaces.
    [Answer: See figure 3.22.]

Figure 3.22: Necklaces with One Red, Two Blue and Three Purple beads



57. How many necklaces are there up to rotations and reflections, with seven beads of two possible colors? [Answer: Our group G is D_7. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 2^1$. Each of the seven reflections have $|X^g| = 2^4$. For the identity $X^g = 2^7$. We then get $\frac{1}{|G|}(6 \times 2^1 + 7 \times 2^4 + 2^7) = \frac{252}{14} = 18$. Thus there are eighteen distinct necklaces.]

Figure 3.23: Necklaces with Seven Beads of Two Colors



58. Draw these necklaces.
[Answer: See figure 3.23.]

59. How many necklaces are there, up to rotations and reflections, with seven beads of three possible colors? [Answer: Our group G is D_7. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 3^1$. Each of the seven reflections have $|X^g| = 3^4$. For the identity $X^g = 3^7$. We then get $\frac{1}{|G|}(6 \times 3^1 + 7 \times 3^4 + 3^7) = \frac{2772}{14} = 198$. Thus there are 198 distinct necklaces.]

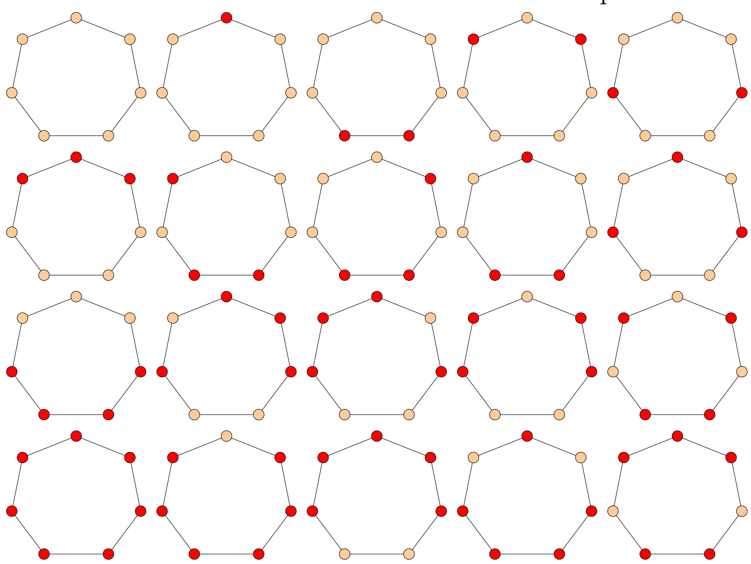
60. How many necklaces are there, up to rotations and reflections, with seven beads of four possible colors? [Answer: Our group G is D_7. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 4^1$. Each of the seven reflections have $|X^g| = 4^4$. For the identity $X^g = 4^7$. We then get $\frac{1}{|G|}(6 \times 4^1 + 7 \times 4^4 + 4^7) = \frac{18200}{14} = 1300$. Thus there are 1300 distinct necklaces.]

61. How many necklaces are there, up to rotations and reflections, with seven beads of n possible colors?

[Answer: Our group G is D_7. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = n^1$. Each of the seven reflections have $|X^g| = n^4$. For the identity $X^g = n^7$. We then get $\frac{1}{|G|}(6n^1+7n^4+n^7) = \frac{1}{14}n(n+1)(n^2-n+1)(n^3+6)$.]

62. How many necklaces are there, up to rotations alone, with seven beads of two possible colors?
[Answer: Our group G is $\mathbb{Z}_7$. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 2$. For the identity $X^g = 2^7$. We then get $\frac{1}{|G|}(12 + 128) = \frac{140}{7} = 20$.]

Figure 3.24: Necklaces with Seven Beads of Two Colors up to Rotation Alone



63. Draw these necklaces.
[Answer: See figure 3.24.]

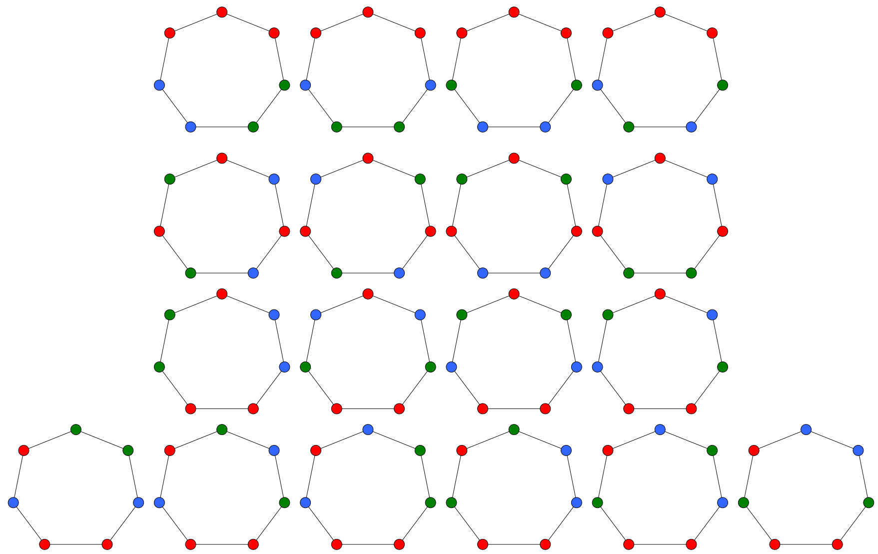
64. How many necklaces are there, up to rotations alone, with seven beads of n possible colors?
[Answer: Our group G is $\mathbb{Z}_7$. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = n^1$. For the identity $X^g = n^7$. We then get $\frac{1}{|G|}(6n^1 + n^7) = \frac{1}{7}n(n^6 + 6)$.]

65. How many seven bead necklaces are there, up to rotations and reflections, with three red, three green and two blue beads?
[Answer: None. If we have three red, three green and two blue beads then we have eight total beads.]

66. How many seven bead necklaces are there, up to rotations and reflections, with three red, two green and two blue beads?
[Answer: Our group G is D_7 but our set X is now just the necklaces with this exact number of beads of each color. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 0$. Each of the seven reflections have $|X^g| = 6$. For the identity $|X^g| = C(7, 2)C(5, 2)C(3, 3) = 210$. We then get $\frac{1}{|G|}(7 \times 6 + 210) = \frac{252}{14} = 18$. Thus there are eighteen distinct necklaces.]

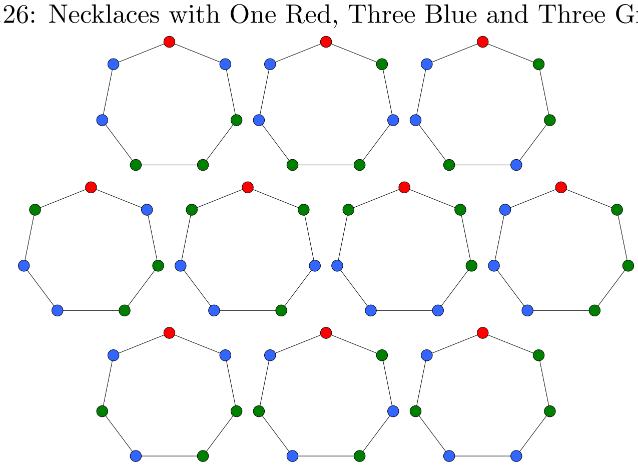
67. Draw these necklaces.
[Answer: See figure 3.25.]

Figure 3.25: Necklaces with Three Red, Two Blue and Two Green Beads



68. How many seven bead necklaces are there, up to rotations and reflections, with one red, three green and three blue beads?
[Answer: Our group G is D_7. For r, r^2, r^3, r^4, r^5 and r^6, $|X^g| = 0$. Each of the seven reflections have $|X^g| = 0$. For the identity $|X^g| = C(7,3)C(4,3)C(1,1) = 140$. We then get $\frac{1}{|G|}(140) = \frac{140}{14} = 10$. Thus there are ten distinct necklaces.]

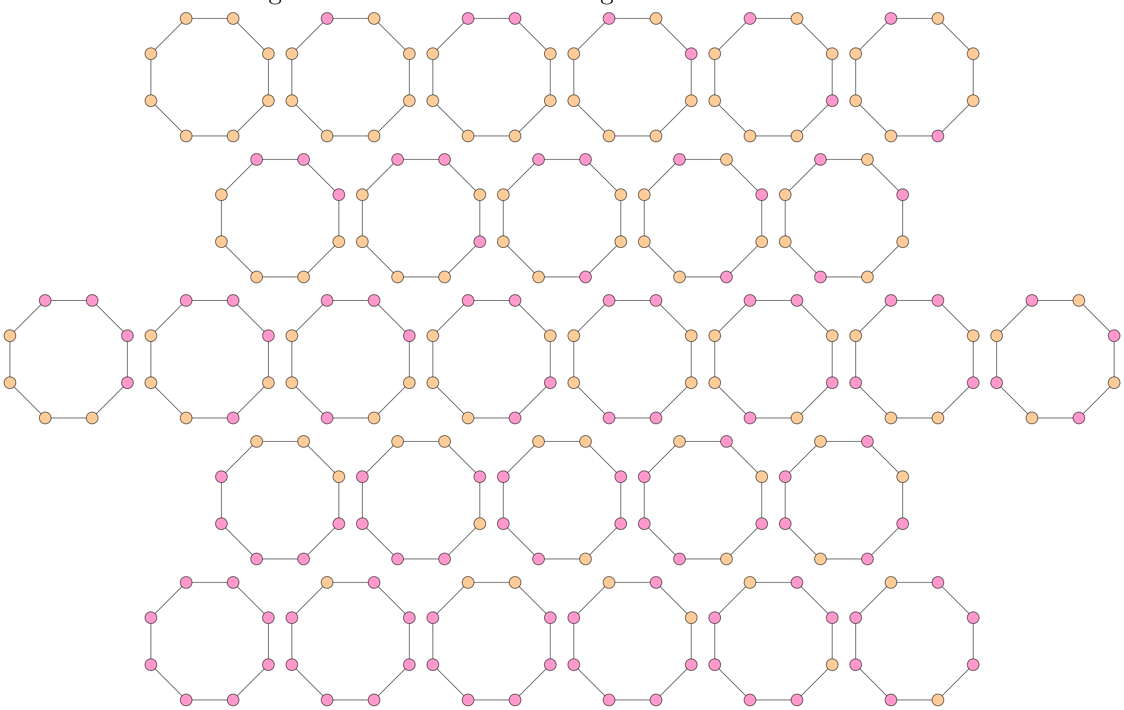
Figure 3.26: Necklaces with One Red, Three Blue and Three Green Beads



69. Draw these necklaces.
[Answer: See figure 3.26.]

70. How many necklaces are there up to rotations and reflections, with eight beads of two possible colors?
[Answer: Our group G is D_8. For r, r^3, r^5 and r^7, $|X^g| = 2^1$. For r^2 and r^6 we get $|X^g| = 2^2$. For r^4 we have $|X^g| = 2^4$. For the four reflections about an axis between two beads we get $|X^g| = 2^4$. For the four reflections about an axis through two beads we have $|X^g| = 2^5$. For the identity $X^g = 2^8$. We then get $\frac{1}{|G|}(4 \times 2^1 + 2 \times 2^2 + 5 \times 2^4 + 4 \times 2^5 + 2^8) = \frac{480}{16} = 30$. Thus there are thirty distinct necklaces.]

Figure 3.27: Necklaces with Eight Beads of Two Colors



71. Draw these necklaces.
[Answer: See figure 3.27.]

72. How many necklaces are there up to rotations and reflections, with eight beads of three possible colors?
[Answer: Our group G is D_8. For r, r^3, r^5 and r^7, $|X^g| = 3^1$. For r^2 and r^6 we get $|X^g| = 3^2$. For r^4 we have $|X^g| = 3^4$. For the four reflections about an axis between two beads we get $|X^g| = 3^4$. For the four reflections about an axis through two beads we have $|X^g| = 3^5$. For the identity $X^g = 3^8$. We then get $\frac{1}{|G|}(4 \times 3^1 + 2 \times 3^2 + 5 \times 3^4 + 4 \times 3^5 + 3^8) = \frac{7968}{16} = 498$. Thus there are 498 distinct necklaces.]

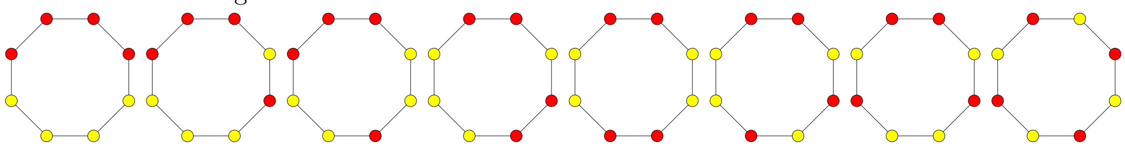
73. How many necklaces are there up to rotations and reflections, with eight beads of n possible colors?
[Answer: Our group G is D_8. For r, r^3, r^5 and r^7, $|X^g| = n^1$. For r^2 and r^6 we get $|X^g| = n^2$. For r^4 we have $|X^g| = n^4$. For the four reflections about an axis between two beads we get $|X^g| = n^4$. For the four reflections about an axis through two beads we have $|X^g| = n^5$. or the identity $X^g = n^8$. We then get $\frac{1}{|G|}(4 \times n^1 + 2 \times n^2 + 5 \times n^4 + 4 \times n^5 + n^8) = \frac{1}{16}n(n+1)(n^6 - n^5 + n^4 + 3n^3 + 2n^2 - 2n + 4)$.]

74. How many necklaces are there up to rotations alone, with eight beads of two colors?
[Answer: Our group G is $\mathbb{Z}_8$. For r, r^3, r^5 and r^7, $|X^g| = 2$. For r^2 and r^6 we get $|X^g| = 4$. For r^4 we have $|X^g| = 16$. The identity has $X^g = 256$. We then get $\frac{1}{|G|}(4 \times 2 + 2 \times 4 + 1 \times 16 + 256) = \frac{288}{8} = 36$.]

75. How many necklaces are there up to rotations alone, with eight beads of n possible colors?
[Answer: Our group G is $\mathbb{Z}_8$. For r, r^3, r^5 and r^7, $|X^g| = n^1$. For r^2 and r^6 we get $|X^g| = n^2$. For r^4 we have $|X^g| = n^4$. The identity has $X^g = n^8$. We then get $\frac{1}{|G|}(4 \times n^1 + 2 \times n^2 + 1 \times n^4 + n^8) = \frac{1}{8}n(n+1)(n^6 - n^5 + n^4 - n^3 + 2n^2 - 2n + 4)$.]

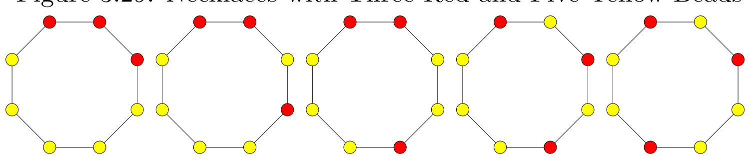
76. How many necklaces are there up to rotations and reflections, with four red and four yellow beads?
[Answer: Our group G is D_8. For r, r^3, r^5, and r^7, $|X^g| = 0$. For r^2 and r^6, $|X^g| = 2$. For r^4 we have $|X^g| = C(4,2) = 6$. For the eight reflections about any axis, we get $|X^g| = C(4,2) = 6$. For the identity $X^g = C(8,4) = 70$. We then get $\frac{1}{|G|}(2 \times 2 + 9 \times 6 + 70) = \frac{128}{16} = 8$.]

Figure 3.28: Necklaces with Four Red and Four Yellow Beads



77. How many necklaces are there up to rotations and reflections, with three red and five yellow beads?
[Answer: Our group G is D_8. For the four reflections about a line between two beads we have $|X^g| = 0$. For the four reflections about an axis through two beads, we get $|X^g| = 6$. For the identity $X^g = C(8,3)C(5,5) = 56$.We then get $\frac{1}{|G|}(4 \times 6 + 56) = \frac{80}{16} = 5$.]

Figure 3.29: Necklaces with Three Red and Five Yellow Beads



78. Draw these necklaces.
[Answer: See figure 3.29.]

79. How many necklaces are there up to rotations and reflections, with two red, three orange and three yellow beads?
[Answer: Our group G is D_8. For any rotation, $|X^g| = 0$. For the four reflections about a line between two beads we have $|X^g| = 0$. For the four reflections about an axis through two beads, we get $|X^g| = 12$. For the identity $X^g = C(8,2)C(6,3)C(3,3) = 560$. We then get $\frac{1}{|G|}(4 \times 12 + 560) = \frac{608}{16} = 38$.]

80. How many necklaces are there up to rotations and reflections, with two beads each in red, yellow, green and blue?
[Answer: Our group G is D_8. For r, r^2, r^3, r^5, r^6 and r^7, $|X^g| = 0$. For r^4 we have $|X^g| = 4! = 24$. For the four reflections about an axis between two beads we get $|X^g| = 4! = 24$. For the four reflections about an

axis through two beads we also have $|X^g| = 4! = 4$. For the identity $X^g = C(8,2)C(6,2)C(4,2)C(2,2) = 2520$. We then get $\frac{1}{|G|}(9 \times 24 + 2520) = \frac{2736}{16} = 171$.]

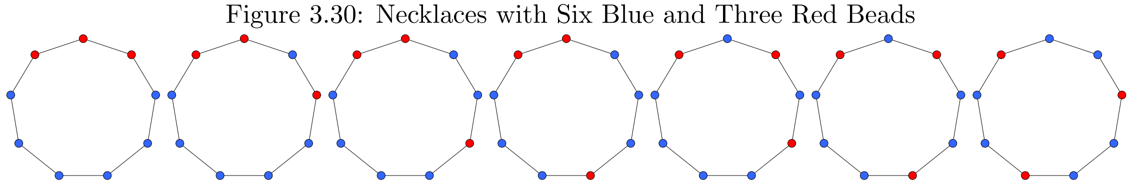
81. How many necklaces are there up to rotations and reflections, with nine beads of two possible colors?
[Answer: Our group G is D_9. For r, r^2, r^4, r^5, r^7 and r^8, $|X^g| = 2^1$. For r^3 and r^6 we get $|X^g| = 2^3$. For any reflection we get $|X^g| = 2^5$. For the identity $|X^g| = 2^9$. We then get $\frac{1}{|G|}(6\times 2^1 + 2\times 2^3 + 9\times 2^5 + 2^9) = \frac{828}{18} = 46$. Thus there are thirty distinct necklaces.]

82. How many necklaces are there up to rotations and reflections, with nine beads of three possible colors?
[Answer: Our group G is D_9. For r, r^2, r^4, r^5, r^7 and r^8, $|X^g| = 3^1$. For r^3 and r^6 we get $|X^g| = 3^3$. For any reflection we get $|X^g| = 3^5$. For the identity $|X^g| = 3^9$. We then get $\frac{1}{|G|}(6\times 3^1 + 2\times 3^3 + 9\times 3^5 + 3^9) = \frac{21942}{18} = 1219$. Thus there are thirty distinct necklaces.]

83. How many necklaces are there up to rotations and reflections, with nine beads of n possible colors?
[Answer: Our group G is D_9. For r, r^2, r^4, r^5, r^7 and r^8, $|X^g| = n^1$. For r^3 and r^6 we get $|X^g| = n^3$. For any reflection we get $|X^g| = n^5$. For the identity $|X^g| = n^9$. We then get $\frac{1}{|G|}(6\times n^1 + 2\times n^3 + 9\times n^5 + n^9) = \frac{1}{18}n(n^8 + 9n^4 + 2n^2 + 6)$.]

84. How many necklaces are there up to rotations and reflections, with six blue, and three red beads?
[Answer: Our group G is D_9. For r, r^2, r^4, r^5, r^7 and r^8, $|X^g| = 0$. For r^3 and r^6 we get $|X^g| = 3$. For any reflection we get $|X^g| = 4$. For the identity $|X^g| = C(9,6) = 84$. We then get $\frac{1}{|G|}(2\times 3 + 9\times 4 + 84) = \frac{126}{18} = 7$. Thus there are seven distinct necklaces.]

Figure 3.30: Necklaces with Six Blue and Three Red Beads



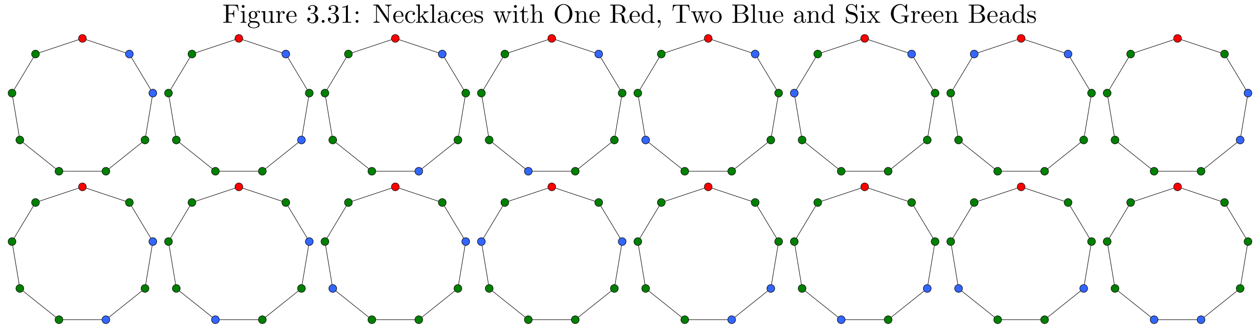
85. Draw these necklaces.
[Answer: See figure 3.30.]

86. How many necklaces are there up to rotations and reflections, with three red, three blue and three green beads?
[Answer: Our group G is D_9. For r, r^2, r^4, r^5, r^7 and r^8, $|X^g| = 0$. For r^3 and r^6 we get $|X^g| = 6$. For any reflection we get $|X^g| = 0$. For the identity $|X^g| = C(9,3)C(6,3)C(3,3) = 1680$. We then get $\frac{1}{|G|}(2 \times 6 + 1680) = \frac{21942}{18} = 94$. Thus there are ninety-four distinct necklaces.]

87. How many necklaces are there up to rotations and reflections, with two red, three blue and four green beads?
[Answer: Our group G is D_9. For all rotations, $|X^g| = 0$. For any reflection we get $|X^g| = 12$. For the identity $|X^g| = C(9,4)C(5,3)C(2,2) = 1260$. We then get $\frac{1}{|G|}(9 \times 12 + 1260) = \frac{1368}{18} = 76$. Thus there are seventy-six distinct necklaces.]

88. How many necklaces are there up to rotations and reflections, with one red, two blue and six green beads?
[Answer: Our group G is D_9. For all rotations, $|X^g| = 0$. For any reflection we get $|X^g| = 4$. For the identity $|X^g| = C(9,2)C(7,6)C(1,1) = 252$. We then get $\frac{1}{|G|}(9 \times 4 + 252) = \frac{288}{18} = 16$. Thus there are sixteen distinct necklaces.]

Figure 3.31: Necklaces with One Red, Two Blue and Six Green Beads

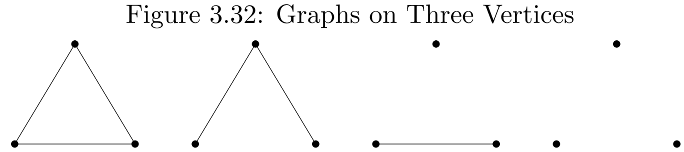


89. Draw these necklaces.
[Answer: See figure 3.31.]

3.2 Graphs

1. How many graphs are there up to isomorphism on three vertices?
[Answer: Our group G is S_3. For any of the three 2-cycles g, $|X^g| = 2^2 = 4$. For either of the two 3-cycles g, $|X^g| = 2^1 = 2$. For the identity $|X^g| = 2^3 = 8$. We then get $\frac{1}{|G|}(3\times4+2\times2+1\times8) = \frac{24}{6} = 4$. Thus there are four graphs.]

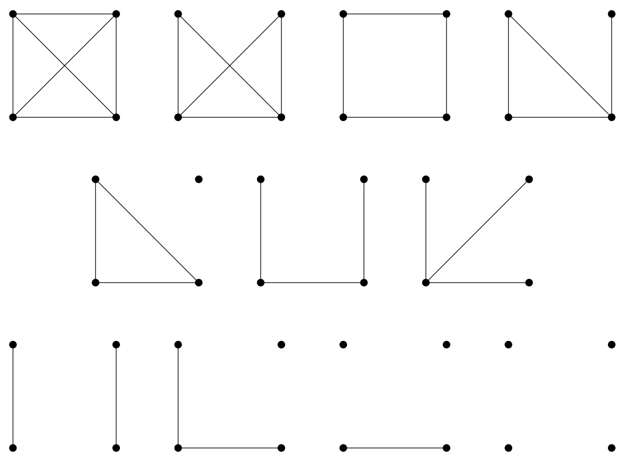
Figure 3.32: Graphs on Three Vertices



2. Draw these graphs.
[Answer: See figure 3.32.]

3. How many graphs are there up to isomorphism on four vertices?
[Answer: Our group G is S_4. For any of the six 2-cycles g, $|X^g| = 2^4$. For any of the three pairs of disjoint 2-cycles we get $|X^g| = 2^4$. For either of the eight 3-cycles g, $|X^g| = 2^2$. For the six four cycles we get $|X^g| = 2^2$. For the identity $|X^g| = 2^6$. We then get $\frac{1}{|G|}(6\times2^4+3\times2^4+8\times2^2+6\times2^2+2^6) = \frac{264}{24} = 11$. Thus there are eleven graphs.]

Figure 3.33: Graphs on Four Vertices



4. Draw these graphs.
[Answer: See figure 3.33.]

5. How many graphs are there up to isomorphism on five vertices?
[Answer: Our group G is S_5. For the twenty-four 5-cycles we get $|X^g| = 2^2$. For the twenty combination 2-cycle with 3-cycle we get $|X^g| = 2^3$. For the thirty 4-cycles we get $|X^g| = 2^3$. For the fifteen pairs of 2-cycles we get $|X^g| = 2^6$. For the twenty 3-cycles we get $|X^g| = 2^4$. For the ten 2-cycles we get $|X^g| = 2^7$. Finally the identity gives us $|X^g| = 2^10$. Thus we get $\frac{1}{120}(24 \times (2^2) + 20 \times (2^3) + 30 \times (2^3) + 15 \times (2^6) + 20 \times (2^4) + 10 \times (2^7) + 2^10) = \frac{4080}{120} = 34$. Thus we get thirty-four graphs.]

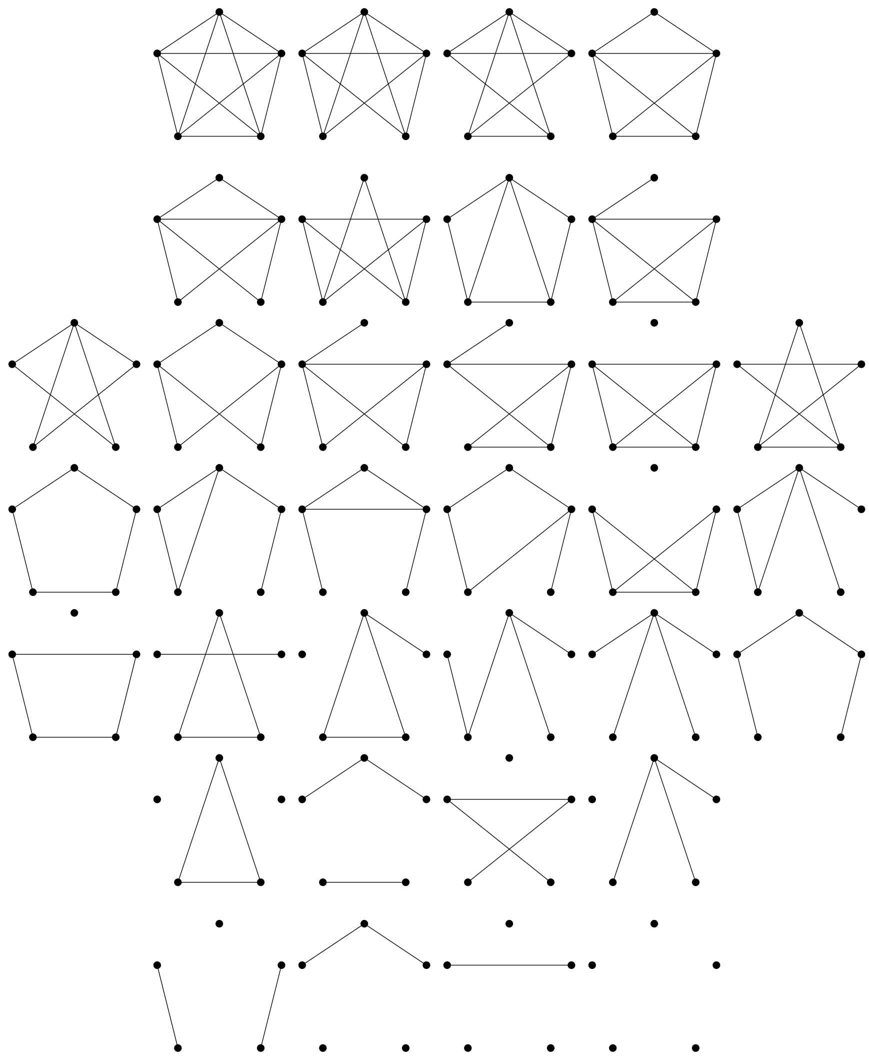
6. Draw these graphs.
[Answer: See figure 3.34.]

7. How many graphs are there up to isomorphism on six vertices?
[Answer: Our group G is S_6. For the 120 6-cycles we get $|X^g| = 2^3$. For the 40 pairs of 3-cycles we get $|X^g| = 2^5$. For the 90 disjoint 4-cycle, 2-cycle pairings we get $|X^g| = 2^5$. For the 15 sets of three disjoint two cycles, $|X^g| = 2^9$.. For the 144 5-cycles we have $|X^g| = 2^3$.. For the 120 disjoint 3-cycle, 2-cycle pairings we get $|X^g| = 2^5$. For the 90 4-cycle pairings we get $|X^g| = 2^5$. For the 45 disjoint two 2-cycle pairings we get $|X^g| = 2^9$. For the forty 3-cycles we get $|X^g| = 2^7$. For the fifteen 2-cycles we get $|X^g| = 2^{11}$. Finally we get $|X^g| = 2^15$ for our identity. We get $\frac{1}{720}(120 \times 2^3 + 40 \times 2^5 + 90 \times 2^5 + 15 \times 2^9 + 144 \times 2^3 + 120 \times 2^5 + 90 \times 2^5 + 45 \times 2^9 + 40 \times 2^7 + 15 \times 2^{11} + 2^{15}) = \frac{112320}{720} = 156$. Thus we get 156 graphs.]

8. How many relations are there on the set $A = \{1, 2\}$, up to possible relabeling? Note that a relation can be seen as either a subset of $\mathcal{P}(A)$ or as a directed graph with possible loops.
[Answer: We take the directed graph approach. The group S_2 acts on this set. For $g = (1, 2)$ we get $|X^g| = 2^2$ and for e we get $|X^g| = 2^4$. Thus we have $\frac{4+16}{2} = 10$ possible relations.

9. Draw these relations.
[Answer: See figure 3.35.]

Figure 3.34: Graphs on Five Vertices



10. How many symmetric relations are there on the set $A = \{1, 2\}$, up to possible relabeling? Note that a symmetic relation can be seen as a graph with possible loops.
[Answer: The group S_2 acts on this set. For $g = (1, 2)$ we get $|X^g| = 2^2$ and for e we get $|X^g| = 2^3$. Thus we have $\frac{4+8}{2} = 6$ possible relations.

11. Draw these relations.
[Answer: See figure 3.36.]

12. How many relations are there on the set $A = \{1, 2, 3\}$, up to possible relabeling?
[Answer: Start with the set of all directed graphs with possible loops. The group S_3 acts on our set.

Figure 3.35: Relations on a Set with Two Elements

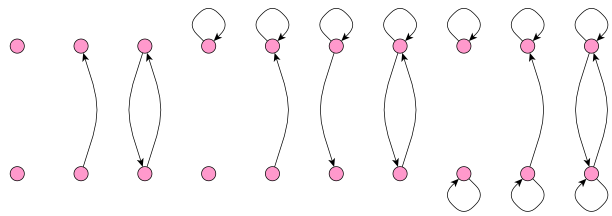
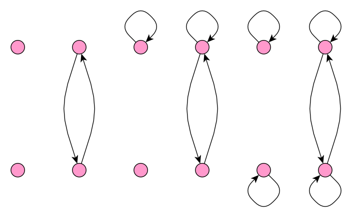


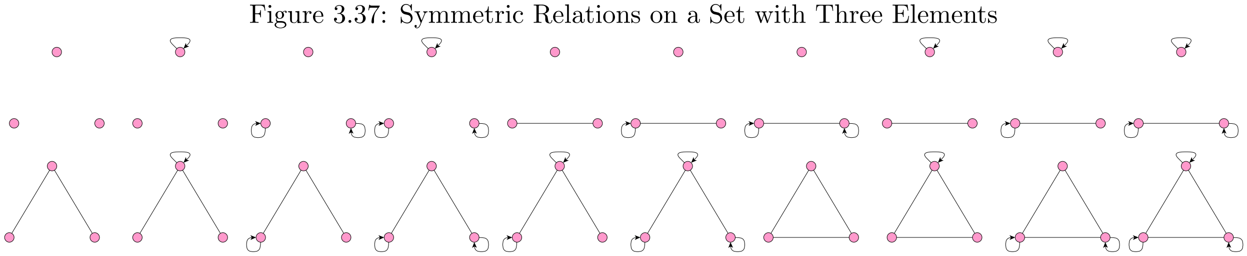
Figure 3.36: Symmetric Relations on a Set with Two Elements



For 3-cycles $|X^g| = 2^3$ For 2-cycles we get $|X^g| = 2^5$ and for e we get $|X^g| = 2^9$. Thus we have $\frac{1}{|G|}(2 \times 2^3 + 3 \times 2^5 + 2^9) = \frac{624}{6} = 104$ possible relations.

13. How many symmetric relations are there on the set $A = \{1, 2, 3\}$, up to possible relabeling?
[Answer: Let X be the set of all graphs with possible loops. The group S_3 acts on our set. For 3-cycles $|X^g| = 2^2$ For 2-cycles we get $|X^g| = 2^4$ and for e we get $|X^g| = 2^6$. Thus we have $\frac{1}{|G|}(2 \times 2^2 + 3 \times 2^4 + 2^6) = \frac{120}{6} = 20$ possible relations.

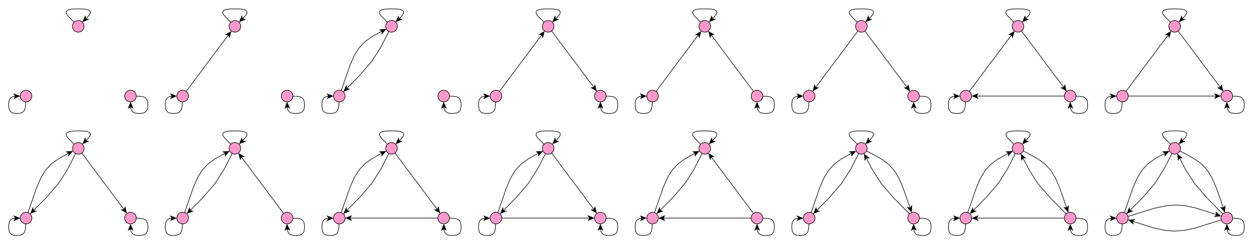
Figure 3.37: Symmetric Relations on a Set with Three Elements



14. Draw these relations.
[Answer: See figure 3.37.]

15. How many reflexive relations are there on the set $A = \{1, 2, 3\}$, up to possible relabeling?
[Answer: Let X be the set of all directed graphs with loops at every vertex.. The group S_3 acts on our set. For 3-cycles $|X^g| = 2^2$ For 2-cycles we get $|X^g| = 2^3$ and for e we get $|X^g| = 2^6$. Thus we have $\frac{1}{|G|}(2 \times 2^2 + 3 \times 2^3 + 2^6) = \frac{96}{6} = 16$ possible relations.

Figure 3.38: Reflexive Relations on a Set with Three Elements

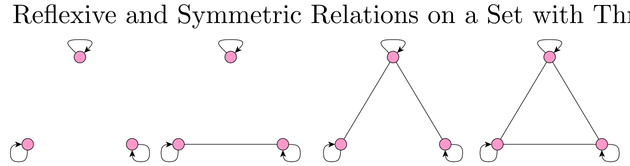


16. Draw these relations.
[Answer: See figure 3.38.]

17. How many symmetric, reflexive relations are there on the set $A = \{1, 2, 3\}$, up to possible relabeling?
[Answer: Let X be the set of all graphs with loops at every vertex. The group S_3 acts on our set. For 3-cycles $|X^g| = 2^1$ For 2-cycles we get $|X^g| = 2^2$ and for e we get $|X^g| = 2^3$. Thus we have $\frac{1}{|G|}(2 \times 2^1 + 3 \times 2^2 + 2^3) = \frac{24}{6} = 4$ possible relations.

18. Draw these relations.
[Answer: See figure 3.39.]

Figure 3.39: Reflexive and Symmetric Relations on a Set with Three Elements



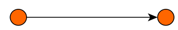
19. How many relations are there on the set $A = \{1, 2, 3, 4\}$, up to possible relabeling?
[Answer: Start with the set of all directed graphs with possible loops. The group S_4 acts on our set. For 4-cycles $|X^g| = 2^4$. For pairs of disjoint 2-cycles we have $|X^g| = 2^8$ For 3-cycles $|X^g| = 2^6$ For 2-cycles we get $|X^g| = 2^{10}$ and for e we get $|X^g| = 2^{16}$. Thus we have $\frac{1}{|G|}(6\times 2^4+3\times 2^8+8\times 2^6+6\times 2^{10}+2^{16}) = \frac{73056}{24} = 3044$ possible relations.

20. A tournament is a directed graph with no loops, where every pair of vertices has an edge in exactly one of the two directions. How many tournaments are there, up to relabeling, on the set $\{1, 2\}$?
[Answer: We have an S_2 action with $|X^g| = 0$ for $g = (1, 2)$ and $|X^g| = 2$ for the identity. This gives us $\frac{1}{2}(2) = 1$.

21. Draw this tournaments.
[Answer: See figure 3.40.]

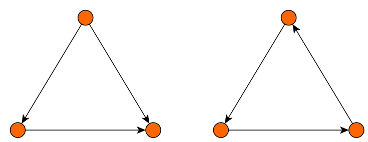
22. How many tournaments are there, up to relabeling, on the set $\{1, 2, 3\}$?
[Answer: We have an S_3 action with $|X^g| = 0$ for any two-cycle, $|X^g| = 2$ for either of the two 3-cycles, and $|X^g| = 2^3$ for the identity. This gives us $\frac{1}{6}(2 \times 2 + 8) = 2$.

Figure 3.40: The Tournament on Two Vertices



23. Draw these tournaments.
[Answer: See figure 3.41.]

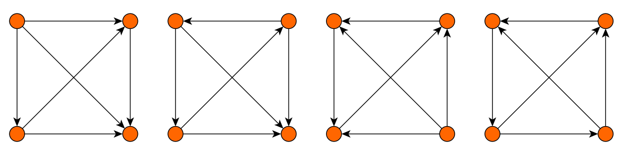
Figure 3.41: Tournaments on Three Vertices



24. How many tournaments are there, up to relabeling, on the set $\{1, 2, 3, 4\}$?
[Answer: We have an S_4 action. We still get $|X^g| = 0$ for any 2-cycle or pairs of two 2-cycles. $|X^g| = 4$ for any of the eight 3-cycles. and $|X^g| = 2^6$ for the identity. This gives us $\frac{1}{24}(8 \times 4 + 64) = 4$.

25. Draw these tournaments.
[Answer: See figure 3.42.]

Figure 3.42: Tournaments on Four Vertices



26. How many tournaments are there, up to relabeling, on the set $\{1, 2, 3, 4, 5\}$?
[Answer: We have an S_5 action. We get $|X^g| = 0$ for any 2-cycle, pairs of two 2-cycles, or 3-cycles paired with 2-cycles. $|X^g| = 2^4$ for any of the twenty 3-cycles. We have $|X^g| = 0$ for any of the thirty 4-cycles and $|X^g| = 2$ for each of the twenty-four 5-cycles. We get $|X^g| = 2^1 0$ for the identity. This gives us $\frac{1}{120}(20 \times 2^4 + 24 \times 2 + 1024) = 4$.

27. Draw these tournaments.
[Answer: See figure 3.43.]

3.3 Platonic Solids

1. How many distinct ways can we color the corners of a tetrahedon with two colors?
[Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 2^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 2^2$. For the identity we have $|X^g| = 2^4$. This gives us $\frac{1}{|G|}(8 \times 2^2 + 3 \times 2^2 + 2^4) = \frac{60}{12} = 5$. Thus there are five tetrahedra.]

Figure 3.43: Tournaments on Five Vertices

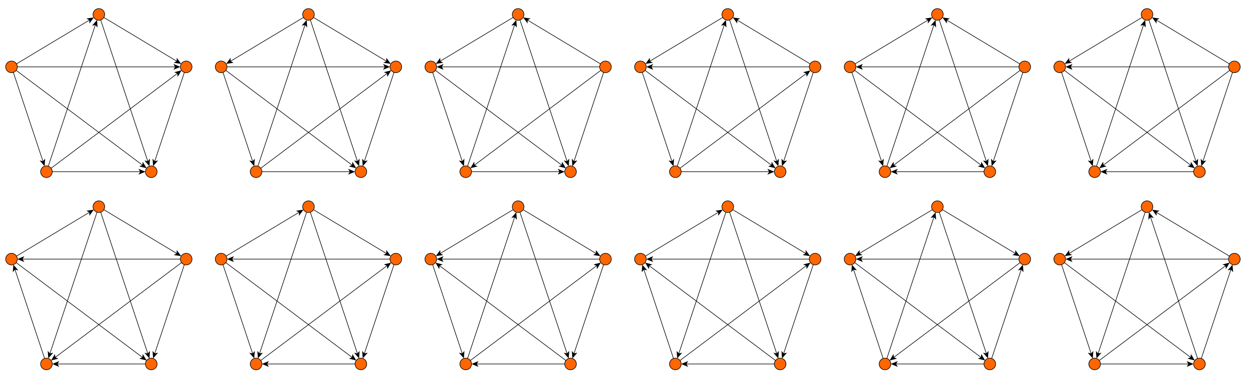
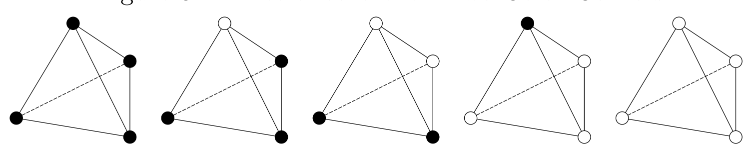


Figure 3.44: Tetrahedra with Two Color Corners



2. Draw these tetrahedra.
   [Answer: See figure 3.44.]

3. How many distinct ways can we color the corners of a tetrahedon with three colors?
   [Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 3^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 3^2$. For the identity we have $|X^g| = 3^4$. This gives us $\frac{1}{|G|}(8 \times 3^2 + 3 \times 3^2 + 3^4) = \frac{180}{12} = 15$. Thus there are fifteen tetrahedra.]

4. Draw these tetrahedra.
   [Answer: See figure 3.45.]

5. How many distinct ways can we color the corners of a tetrahedon with four colors?
   [Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 4^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 4^2$. For the identity we have $|X^g| = 4^4$. This gives us $\frac{1}{|G|}(8 \times 4^2 + 3 \times 4^2 + 4^4) = \frac{432}{12} = 36$. Thus there are fifteen tetrahedra.]

6. Draw these tetrahedra.
   [Answer: See figure 3.46.]

7. How many distinct ways can we color the corners of a tetrahedon with n colors?
   [Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = n^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = n^2$. For the identity we have $|X^g| = n^4$. This gives us $\frac{1}{|G|}(8n^2 + 3n^2 + n^4) = \frac{n^2}{12}(11 + n^2)$.]

Figure 3.45: Tetrahedra with Three Color Corners

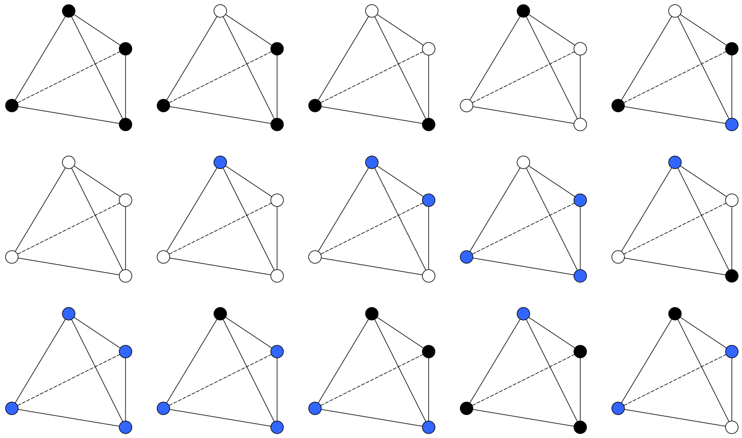
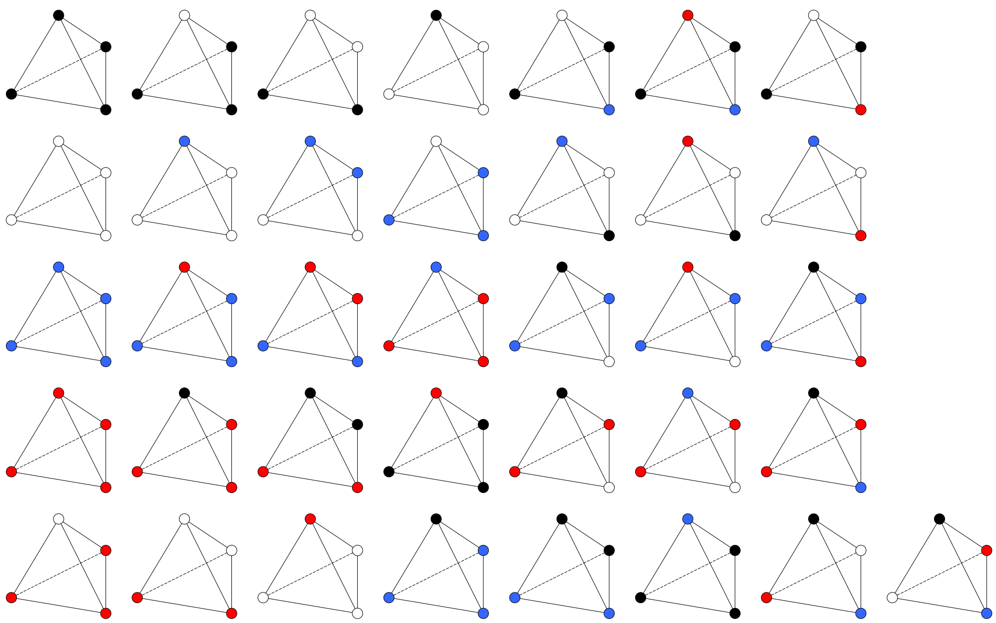


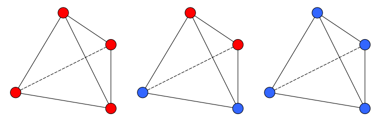
Figure 3.46: Tetrahedra with Four Color Corners



8. How many distinct ways can we color the corners of a tetrahedon with two colors so that no color appears only once?
[Answer: Our group G is A_4. Our set X is the collection of tetrahedral corner colorings in two colors so each color appears more than once. For the eight 120 degree rotations about a corner we have $|X^g| = 2$ as the corner we are rotating about is forced to match the other three. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 4$. For the identity we have $|X^g| = 2 + 3 \times 2 = 8$. To

see this break things down into the two cases all of one color, and the two cases with two colors, each corresponding to a choice of opposite edges. This gives us $\frac{1}{|G|}(8 \times 2 + 3 \times 4 + 8) = \frac{36}{12} = 3$.]

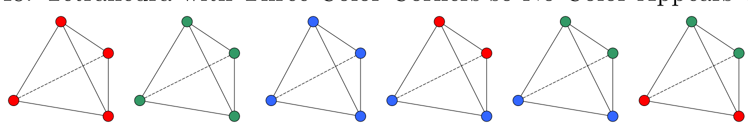
Figure 3.47: Tetrahedra with Two Color Corners so No Color Appears Only Once



9. Draw these tetrahedra.
   [Answer: See figure 3.47.]

10. How many distinct ways can we color the corners of a tetrahedon with three colors so that no color appears only once?
    [Answer: Our group G is A_4. Our set X is the collection of tetrahedral corner colorings in two colors so each color appears more than once. For the eight 120 degree rotations about a corner we have $|X^g| = 3$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 9$. For the identity we have $|X^g| = 3 + 18$. This gives us $\frac{1}{|G|}(8 \times 3 + 3 \times 9 + 21) = \frac{72}{12} = 6$.]

Figure 3.48: Tetrahedra with Three Color Corners so No Color Appears Only Once



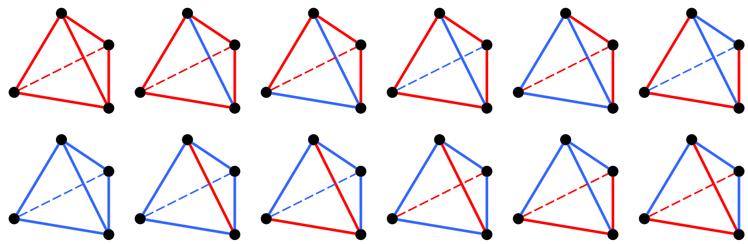
11. Draw these tetrahedra.
    [Answer: See figure 3.48.]

12. How many distinct ways can we color the corners of a tetrahedon with n colors so that no color appears only once?
    [Answer: Our group G is A_4. Our set X is the collection of tetrahedral corner colorings so each color appears more than once. For the eight 120 degree rotations about a corner we have $|X^g| = n$ as the corner we are rotating about is forced to match the other three. For each of the three 180 degree rotations about opposite edges we get $|X^g| = n^2$ since it is forced that no color can appear only once if the coloring is fixed under that rotation. For the identity we have $|X^g| = n + 3n(n-1)$. To get this, we took the colorings of all one color, and then added the possibilities for two colors. This gives us $\frac{1}{|G|}(8n + 3n^2 + 3n^2 - 2n) = \frac{1}{12}6n(n+1) = \frac{n(n+1)}{2}$. Notice that this gives us the triangular numbers. This is sequence A000217 in the OEIS.]

13. How many distinct ways can we color the edges of a tetrahedon with two colors?
    [Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 2^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 2^4$. For the identity we have $|X^g| = 2^6$. This gives us $\frac{1}{|G|}(8 \times 2^2 + 3 \times 2^4 + 2^6) = \frac{144}{12} = 12$. Thus there are twelve tetrahedra.]

14. Draw these tetrahedra.
    [Answer: See figure 3.49.]

Figure 3.49: Tetrahedra with Two Color Edges



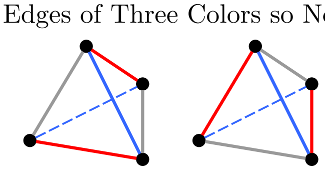
15. How many distinct ways can we color the edges of a tetrahedon with three colors?
[Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 3^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 3^4$. For the identity we have $|X^g| = 3^6$. This gives us $\frac{1}{|G|}(8 \times 3^2 + 3 \times 3^4 + 3^6) = \frac{1044}{12} = 87$. Thus there are eighty-seven ways to color our tetrahedron.]

16. How many distinct ways can we color the edges of a tetrahedon with four colors?
[Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = 4^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 4^4$. For the identity we have $|X^g| = 4^6$. This gives us $\frac{1}{|G|}(8 \times 4^2 + 3 \times 4^4 + 4^6) = \frac{4992}{12} = 416$. Thus there are 416 ways to color our tetrahedron.]

17. How many distinct ways can we color the edges of a tetrahedon with n colors?
[Answer: Our group G is A_4. For the eight 120 degree rotations about a corner we have $|X^g| = n^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = n^4$. For the identity we have $|X^g| = n^6$. This gives us $\frac{1}{|G|}(8n^2 + 3n^4 + n^6) = \frac{n^2}{12}(8 + 3n^2 + n^4)$.]

18. How many distinct ways can we color the edges of a tetrahedon with three colors so that no two adjacent edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in three colors that have no adjacent edges of the same color. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 3! = 6$. For the identity we have $|X^g| = 3! = 6$. This gives us $\frac{1}{|G|}(3 \times 6 + 6) = \frac{24}{12} = 2$. Thus there are only two inequivalent ways to color a tetrahedon in three colors so that no two adjacent edges share a color.]

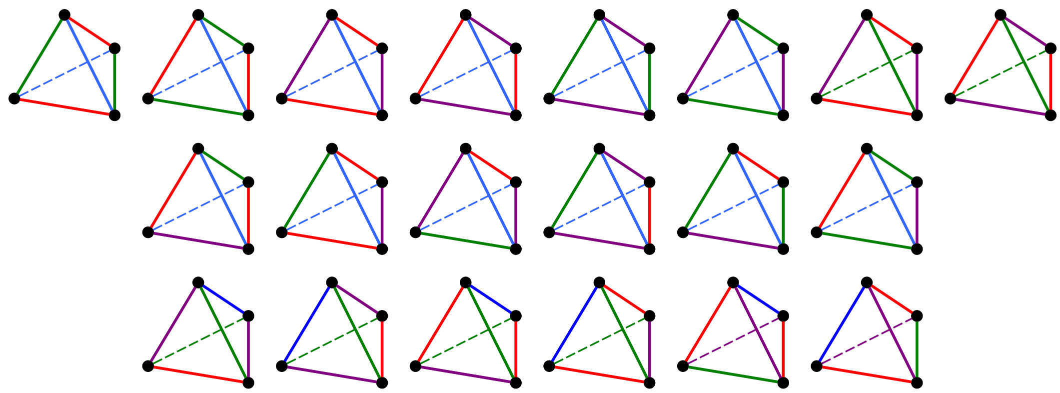
Figure 3.50: Tetrahedra with Edges of Three Colors so No Adjacent Edges Share a Color



19. Draw these tetrahedra.
[Answer: See figure 3.50.]

20. How many distinct ways can we color the edges of a tetrahedon with four colors so that no two adjacent edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in four colors that have no adjacent edges of the same color. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 4\times3\times2\times2 = 48$. For the identity calculation we can add the two separate cases of there being 3 total colors and 4 total colors. We get $|X^g| = 4\times3\times2+3\times4\times3\times2 = 24+72 = 96$. This gives us $\frac{1}{|G|}(3\times48+96) = \frac{240}{12} = 20$. Thus there are twenty inequivalent ways to color a tetrahedon in four colors so that no two adjacent edges share a color.]

Figure 3.51: Tetrahedra with Edges of Four Colors so No Adjacent Edges Share a Color

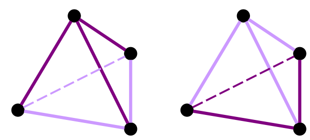


21. Draw these tetrahedra.
[Answer: See figure 3.51.]

22. How many distinct ways can we color the edges of a tetrahedon with n colors so that no two adjacent edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in two colors that have no adjacent edges of the same color. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = n(n-1)(n-2)^2$. For the identity we can break things down into the colorings that have exactly 6,5,4 and then 3 distinct colors. We have $|X^g| = n(n-1)(n-2)(n-3)(n-4)(n-5)+3n(n-1)(n-2)(n-3)(n-4)+3n(n-1)(n-2)(n-3)+n(n-1)(n-2)$. This gives us $\frac{1}{|G|}(3\times n(n-1)(n-2)^2+n(n-1)(n-2)(n-3)(n-4)(n-5)+3n(n-1)(n-2)(n-3)(n-4)+3n(n-1)(n-2)(n-3)+n(n-1)(n-2)) = \frac{1}{12}n(n-1)(n-2)(n^3-9n^2+32n-38)$.]

23. How many distinct ways can we color the edges of a tetrahedon with two colors so that no two opposite edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in two colors that have no opposite edges of the same color. For the eight 120 degree rotations about a corner we have $|X^g| = 2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 0$. For the identity we have $|X^g| = 2^3 = 8$. This gives us $\frac{1}{|G|}(8\times2+8) = \frac{24}{12} = 2$. Thus there are only two inequivalent ways to color a tetrahedon in three colors so that no two opposite edges share a color.]

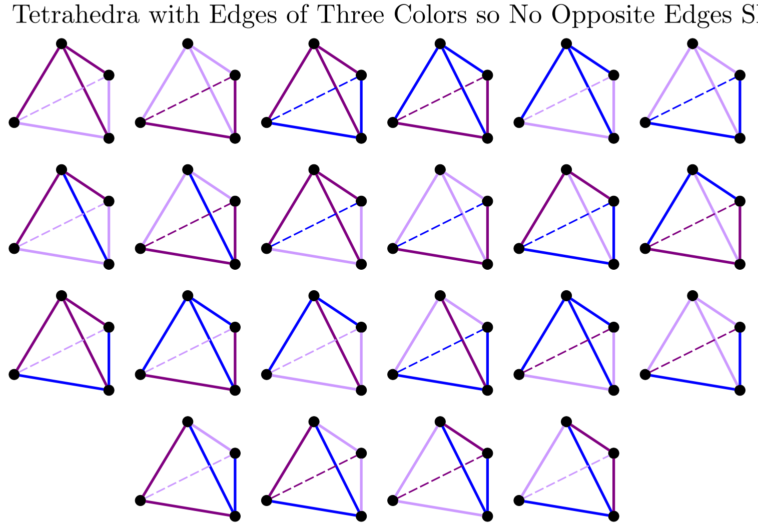
Figure 3.52: Tetrahedra with Edges of Two Colors so No Opposite Edges Share a Color



24. Draw these tetrahedra.
[Answer: See figure 3.52.]

25. How many distinct ways can we color the edges of a tetrahedon with three colors so that no two opposite edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in three colors that have no opposite edges of the same color. For the eight 120 degree rotations about a corner we have $|X^g| = 3 \times 2 = 6$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 0$. For the identity we have $|X^g| = 3^3 \times 2^3$. To see this, think about coloring the three edges about one corner, then consider which choices are left for the remaining three. This gives us $\frac{1}{|G|}(8 \times 6 + 3^3 \times 2^3) = \frac{264}{12} = 22$. Thus there are twenty-two inequivalent ways to color a tetrahedon in three colors so that no two opposite edges share a color.]

Figure 3.53: Tetrahedra with Edges of Three Colors so No Opposite Edges Share a Color



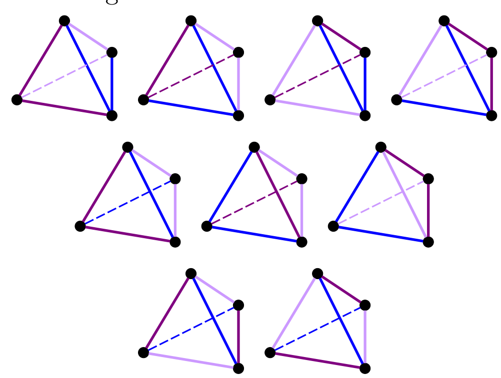
26. Draw these tetrahedra.
[Answer: See figure 3.53.]

27. How many distinct ways can we color the edges of a tetrahedon with n colors so that no two opposite edges share the same color?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in n colors that have no opposite edges of the same color. For the eight 120 degree rotations about a corner

we have $|X^g| = n(n-1)$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 0$. For the identity we have $|X^g| = n^3(n-1)^3$. To see this, again think about coloring the three edges about one corner, then consider which choices are left for the remaining three. This gives us $\frac{1}{|G|}(8n(n-1) + n^3(n-1)^3) = \frac{1}{12}n(n-1)(n^4 - 2n^3 + n^2 + 8)$.]

28. How many distinct ways can we color the edges of a tetrahedon with three colors so that no color appears more than twice?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in three colors with no color appearing three or more times. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 3 \times 2$. For the identity notice that each color must be repeated twice to get $|X^g| = 15 \times 3 \times 2 = 90$. This gives us $\frac{1}{|G|}(3 \times 6 + 90) = \frac{108}{12} = 9$.]

Figure 3.54: Tetrahedra with Edges of Three Colors so No Color Appears More than Twice



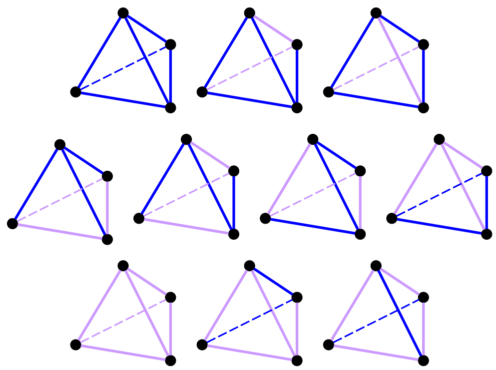
29. Draw these tetrahedra.
[Answer: See figure 3.54.]

30. How many distinct ways can we color the edges of a tetrahedon with n colors so that no color appears more than twice?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in n colors with no color appearing three or more times. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = n(n-1)(n-2)^2$. For the identity we break things down into the cases of no repeated colors through till we get to the case of three repeated colors. This gives us $|X^g| = n(n-1)(n-2)(n-3)(n-4)(n-5) + 15n(n-1)(n-2)(n-3)(n-4) + 45n(n-1)(n-2)(n-3) + 15n(n-1)(n-2)$. We now arrive at a formula of $\frac{1}{|G|}(3 \times n(n-1)(n-2)^2 + n(n-1)(n-2)(n-3)(n-4)(n-5) + 15n(n-1)(n-2)(n-3)(n-4) + 45n(n-1)(n-2)(n-3) + 15n(n-1)(n-2)) = \frac{1}{12}n(n-2)(n-1)(n^3 + 3n^2 - 10n - 6)$.]

31. How many distinct ways can we color the edges of a tetrahedon with n colors so that no color appears only once?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in two colors with each color appearing at least twice. For the eight 120 degree rotations about a corner we have $|X^g| = 4$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 4 + 8 = 12$. To see this,

break things down into the cases where the edges are the same color and the edges are different colors. For the identity we break things down into the 2-6 and 3-3 two color cases and the one color case. This gives us $|X^g| = 15 \times 2 + 10 \times 2 + 2$. We now arrive at a formula of $\frac{1}{|G|}(8 \times 4 + 3 \times 12 + 52) = \frac{120}{12} = 10$. Thus there are exactly ten such colorings.]

Figure 3.55: Tetrahedra with Edges of Two Colors so No Color Appears Only Once



32. Draw these tetrahedra.
[Answer: See figure 3.55.]

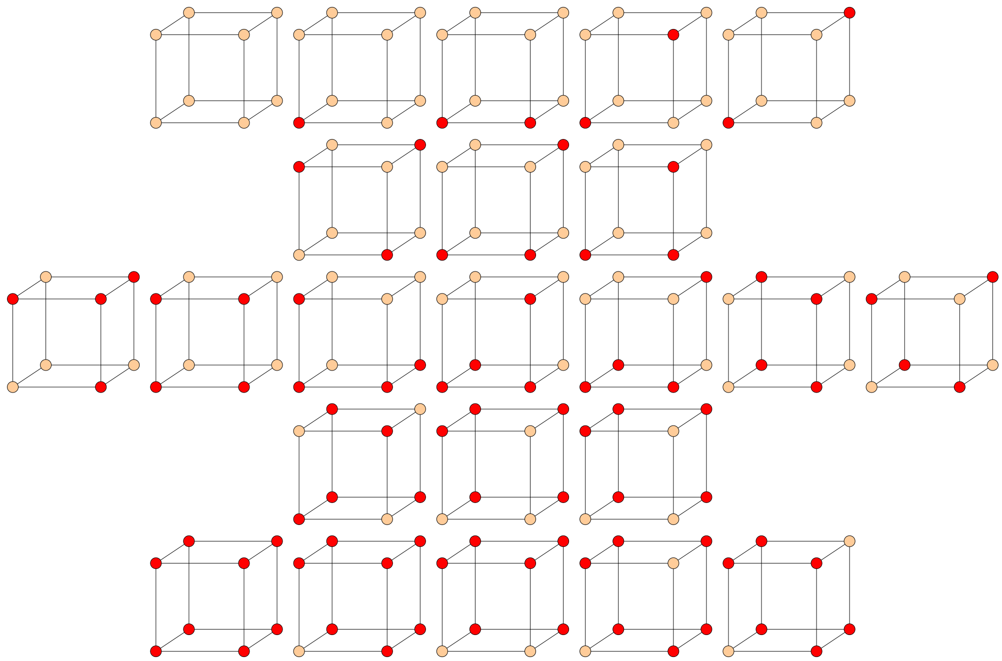
33. How many distinct ways can we color the edges of a tetrahedon with n colors so that no color appears only once?
[Answer: Our group G is A_4. Our set X is the collection of edge colored tetrahedra in n colors with each color appearing at least twice. For the eight 120 degree rotations about a corner we have $|X^g| = n^2$. For each of the three 180 degree rotations about opposite edges we get $|X^g| = 2n(n-1)+n^3$. For the identity we break things down into the three color, two color, and one color cases. This gives us $|X^g| = 15n(n-1)(n-2) + 15n(n-1) + 10n(n-1) + n$. We now arrive at a formula of $\frac{1}{|G|}(8 \times n^2 + 3 \times (2n(n-1)+n^3) + 15n(n-1)(n-2) + 15n(n-1) + 10n(n-1) + n) = \frac{n^2(3n-1)}{2}$. This is sequence A050509 in the OEIS.]

34. How many distinct ways can we color the corners of a cube with two colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 2^2$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 2^4$. For the eight 120 degree rotations about a corner we have $|X^g| = 2^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 2^4$. The identity has $|X^g| = 2^8$. This gives us $\frac{1}{|G|}(6 \times 2^2 + 3 \times 2^4 + 8 \times 2^4 + 6 \times 2^4 + 2^8) = \frac{552}{24} = 23$. Thus there are twenty-three cubes.]

35. Draw these cubes.
[Answer: See figure 3.56.]

36. How many distinct ways can we color the corners of a cube with three colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 3^2$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 3^4$. For the eight 120 degree rotations about a corner we have $|X^g| = 3^4$. For each of the six 180 degree rotations about

Figure 3.56: Cubes with Two Color Corners



opposite edges we get $|X^g| = 3^4$. The identity has $|X^g| = 3^8$. This gives us $\frac{1}{|G|}(6 \times 3^2 + 3 \times 3^4 + 8 \times 3^4 + 6 \times 3^4 + 3^8) = \frac{7992}{24} = 333$. Thus there are twenty-three cubes.]

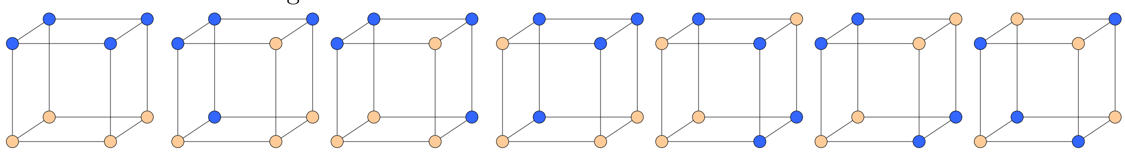
37. How many distinct ways can we color the corners of a cube with four colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 4^2$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 4^4$. For the eight 120 degree rotations about a corner we have $|X^g| = 4^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 4^4$. The identity has $|X^g| = 4^8$. This gives us $\frac{1}{|G|}(6 \times 4^2 + 3 \times 4^4 + 8 \times 4^4 + 6 \times 4^4 + 4^8) = \frac{69984}{24} = 2916$. Thus there are 2916 cubes.]

38. How many distinct ways can we color the corners of a cube with n colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = n^2$. For the three 180 degree rotations about two opposite faces we get $|X^g| = n^4$. For the eight 120 degree rotations about a corner we have $|X^g| = n^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = n^4$. The identity has $|X^g| = n^8$. This gives us $\frac{1}{|G|}(6n^2 + 3n^4 + 8n^4 + 6n^4 + n^8) = \frac{n^2}{24}(6 + 17n^2 + n^6)$.]

39. How many distinct ways can we color the corners of a cube with four tan and four blue corners?
[Answer: Our group G is S_4. Recall that our X now only contains cubes with exactly four colors of each type. For the six ninety degree rotations about two opposite faces we get $|X^g| = 2$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 6$. For the eight 120 degree rotations about a corner we have $|X^g| = 4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 6$.

The identity has $|X^g| = C(8,4) = 70$. This gives us $\frac{1}{|G|}(6 \times 2 + 3 \times 6 + 8 \times 4 + 6 \times 6 + 70) = \frac{168}{24} = 7$. Thus there are seven cubes.]

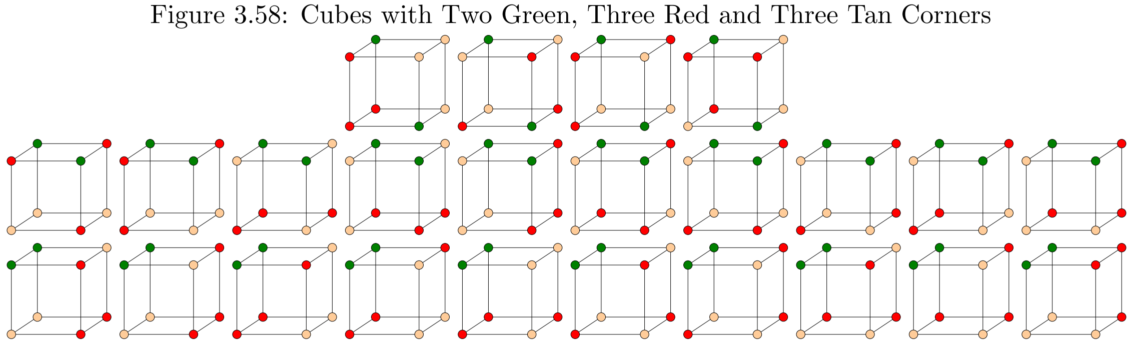
Figure 3.57: Cubes with Four Corners of Each Color



40. Draw these cubes.
[Answer: See figure 3.57.]

41. How many distinct ways can we color the corners of a cube with three tan, three red and two green corners?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 0$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 0$. For the eight 120 degree rotations about a corner we have $|X^g| = 2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 0$. The identity has $|X^g| = C(8,3)C(5,3)C(2,2) = 560$. This gives us $\frac{1}{|G|}(8 \times 2 + 560) = \frac{576}{24} = 24$. Thus there are twenty-four such cubes.]

Figure 3.58: Cubes with Two Green, Three Red and Three Tan Corners



42. Draw these cubes.
[Answer: See figure 3.58.]

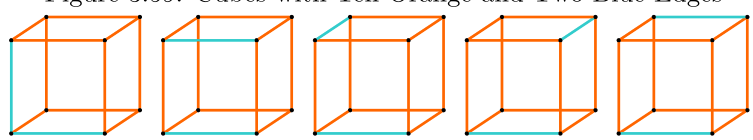
43. How many distinct ways can we color the corners of a cube with two tan, two red, two blue, and two green corners?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 0$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 24$. For the eight 120 degree rotations about a corner we have $|X^g| = 0$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 24$. The identity has $|X^g| = C(8,2)C(6,2)C(4,2)C(2,2) = 2520$. This gives us $\frac{1}{|G|}(3 \times 24 + 6 \times 24 + 2520) = \frac{2736}{24} = 114$. Thus there are 114 cubes.]

44. How many distinct ways can we color the edges of a cube with two blue and ten orange edges?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get

$|X^g| = 2^3$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 2^6$. For the eight 120 degree rotations about a corner we have $|X^g| = 2^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 2^7$. The identity has $|X^g| = 2^{12}$. This gives us $\frac{1}{|G|}(6 \times 2^3 + 3 \times 2^6 + 8 \times 2^4 + 6 \times 2^7 + 2^{12}) = \frac{5232}{24} = 218$. Thus there are 218 cubes.]

45. How many distinct ways can we color the edges of a cube with two blue and ten orange edges?
[Answer: Our group G is S_4. Our set is the $C(12,2)$ cubes with ten orange and two blue edges. For the six ninety degree rotations about two opposite faces and the eight 120 degree rotations about a corner, none of these are fixed so we get $|X^g| = 0$. For the three 180 degree rotations about two opposite faces and the six 180 degree rotations about opposite edges, we get $|X^g| = 6$. The identity has $|X^g| = C(12,2) = 66$. This gives us $\frac{1}{|G|}(9 \times 6 + 66) = \frac{120}{24} = 5$. Thus there are five such cubes.]

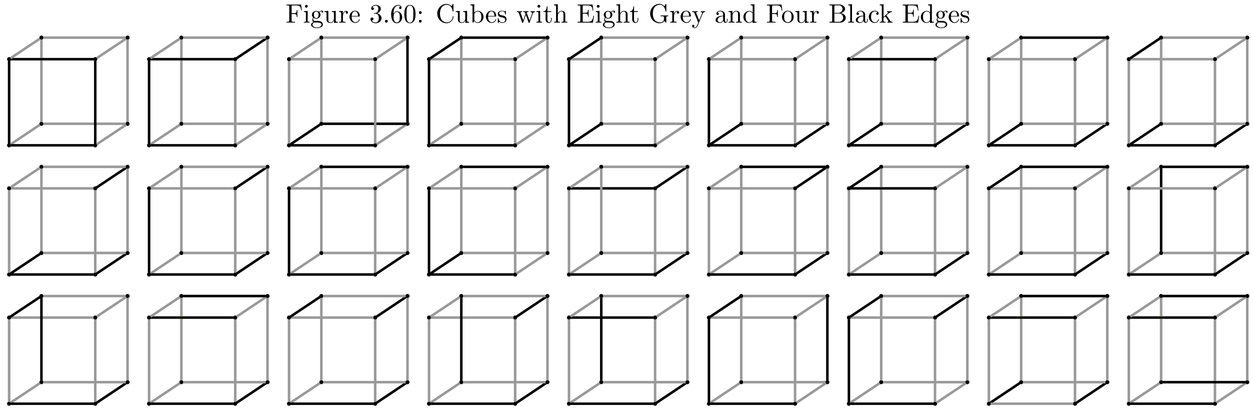
Figure 3.59: Cubes with Ten Orange and Two Blue Edges



46. Draw these cubes.
[Answer: See figure 3.59.]

47. How many distinct ways can we color the edges of a cube four black and eight grey edges?
[Answer: Our group G is S_4. Our set is the $C(12,4)$ cubes with four black and eight grey edges. For the six ninety degree rotations about two opposite faces we have $|X^g| = 3$. For the three 180 degree rotations about two opposite faces we get $|X^g| = C(6,2) = 15$. For the eight 120 degree rotations about a corner, none of these are fixed so we get $|X^g| = 0$. For the six 180 degree rotations about opposite edges, we get $|X^g| = C(6,2) = 15$. The identity has $|X^g| = C(12,4) = 495$. This gives us $\frac{1}{|G|}(6 \times 3 + 9 \times 15 + 495) = \frac{648}{24} = 27$. Thus there are twenty-seven such cubes.]

Figure 3.60: Cubes with Eight Grey and Four Black Edges

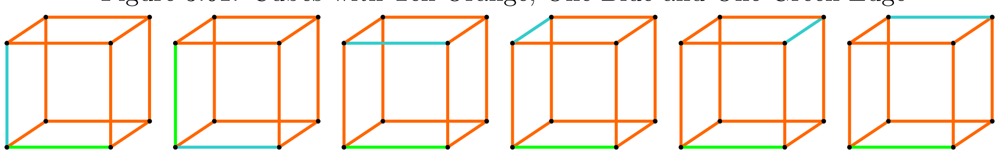


48. Draw these cubes.
[Answer: See figure 3.60.]

49. How many distinct ways can we color the edges of a cube with three colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 3^3$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 3^6$. For the eight 120 degree rotations about a corner we have $|X^g| = 3^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 3^7$. The identity has $|X^g| = 3^{12}$. This gives us $\frac{1}{|G|}(6 \times 3^3 + 3 \times 3^6 + 8 \times 3^4 + 6 \times 3^7 + 3^{12}) = \frac{547560}{24} = 22815$. Thus there are 22815 cubes.]

50. How many distinct ways can we color the edges of a cube with one blue, one green and ten orange edges?
[Answer: Our group G is S_4. Our set is the $C(12,1)C(11,1)$ cubes with ten orange, one green and one blue edge. For the six ninety degree rotations about two opposite faces, the three 180 degree rotations about two opposite faces, and the eight 120 degree rotations about a corner, none of these are fixed so we get $|X^g| = 0$. For the six 180 degree rotations about opposite edges, we get $|X^g| = 2$. The identity has $|X^g| = C(12,1)C(11,1) = 132$. This gives us $\frac{1}{|G|}(6 \times 2 + 132) = \frac{144}{24} = 6$. Thus there are six such cubes.]

Figure 3.61: Cubes with Ten Orange, One Blue and One Green Edge



51. Draw these cubes.
[Answer: See figure 3.61.]

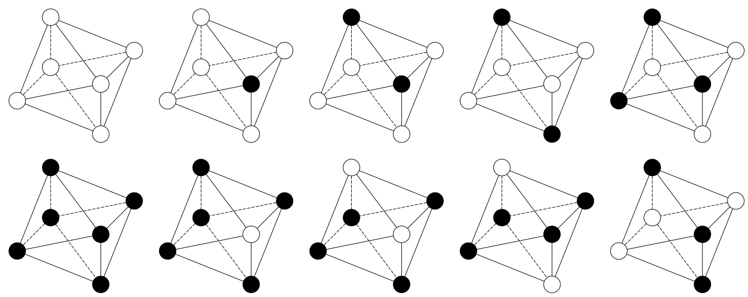
52. How many distinct ways can we color the edges of a cube with four colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = 4^3$. For the three 180 degree rotations about two opposite faces we get $|X^g| = 4^6$. For the eight 120 degree rotations about a corner we have $|X^g| = 4^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 4^7$. The identity has $|X^g| = 4^{12}$. This gives us $\frac{1}{|G|}(6 \times 4^3 + 3 \times 4^6 + 8 \times 4^4 + 6 \times 4^7 + 4^{12}) = \frac{16890240}{24} = 703760$. Thus there are 703760 cubes.]

53. How many distinct ways can we color the edges of a cube with n colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about two opposite faces we get $|X^g| = n^3$. For the three 180 degree rotations about two opposite faces we get $|X^g| = n^6$. For the eight 120 degree rotations about a corner we have $|X^g| = n^4$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = n^7$. The identity has $|X^g| = n^{12}$. This gives us $\frac{1}{|G|}(6n^3 + 3n^6 + 8n^4 + 6n^7 + n^{12}) = \frac{n^3}{24}(6 + 8n + 3n^3 + 6n^4 + n^9)$.]

54. How many distinct ways can we color the corners of an octahedron with two colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about a corner we get $|X^g| = 2^3$. For the three 180 degree rotations about two opposite corners we get $|X^g| = 2^4$. For the eight 120 degree

rotations about opposite faces we have $|X^g| = 2^2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 2^3$. The identity has $|X^g| = 2^6$. This gives us $\frac{1}{|G|}(6\times 2^3+3\times 2^4+8\times 2^2+6\times 2^3+2^6) = \frac{240}{24} = 19$. Thus there are ten octahedra.]

Figure 3.62: Octahedra with Corners of Two Colors



55. Draw these octahedra.
[Answer: See figure 3.62.]

56. How many distinct ways can we color the corners of an octahedron with three colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about a corner we get $|X^g| = 3^3$. For the three 180 degree rotations about two opposite corners we get $|X^g| = 3^4$. For the eight 120 degree rotations about opposite faces we have $|X^g| = 3^2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 3^3$. The identity has $|X^g| = 3^6$. This gives us $\frac{1}{|G|}(6\times 3^3+3\times 3^4+8\times 3^2+6\times 3^3+3^6) = \frac{1368}{24} = 57$. Thus there are fifty-seven octahedra.]

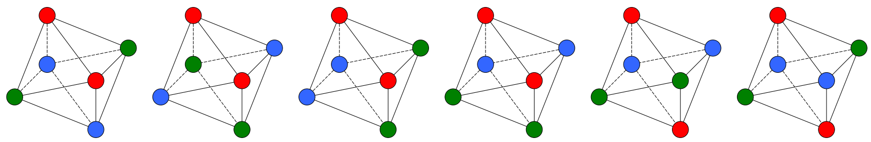
57. How many distinct ways can we color the corners of an octahedron with four colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about a corner we get $|X^g| = 4^3$. For the three 180 degree rotations about two opposite corners we get $|X^g| = 4^4$. For the eight 120 degree rotations about opposite faces we have $|X^g| = 4^2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 4^3$. The identity has $|X^g| = 4^6$. This gives us $\frac{1}{|G|}(6\times 4^3+3\times 4^4+8\times 4^2+6\times 4^3+4^6) = \frac{5760}{24} = 240$. Thus there are 240 octahedra.]

58. How many distinct ways can we color the corners of an octahedron with n colors?
[Answer: Our group G is S_4. For the six ninety degree rotations about a corner we get $|X^g| = n^3$. For the three 180 degree rotations about two opposite corners we get $|X^g| = n^4$. For the eight 120 degree rotations about opposite faces we have $|X^g| = n^2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = n^3$. The identity has $|X^g| = n^6$. This gives us $\frac{1}{|G|}(6n^3 + 3n^4 + 8n^2 + 6n^3 + n^6) = \frac{n^2}{24}(8 + 12n + 3n^2 + n^4)$.]

59. How many distinct ways can we color the corners of an octahedron with two red, two green and two blue corners?
[Answer: Our group G is S_4. Keep in mind X is now the set of octahedron with 2 corners of each of the colors, so there may be nothing in X that is stabilized by certain rotations. For the six ninety degree rotations about a corner we get $|X^g| = 0$. For the three 180 degree rotations about two opposite corners we get $|X^g| = 6$. For the eight 120 degree rotations about opposite faces we have $|X^g| = 0$.

For each of the six 180 degree rotations about opposite edges we get $|X^g| = 6$. The identity has $|X^g| = C(6,2)C(4,2)C(2,2) = 15 \times 6 = 90$. This gives us $\frac{1}{|G|}(3 \times 6 + 6 times 6 + 90) = \frac{144}{24} = 6$. Thus there are six distinct ways to color our octahedron in this way.]

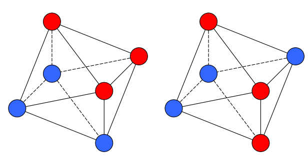
Figure 3.63: Octahedra with Two Corners Each in Three Colors



60. Draw these octahedra.
[Answer: See figure 3.63.]

61. How many distinct ways can we color the corners of an octahedron with three red and three blue corners?
[Answer: Our group G is S_4. Keep in mind X is now the set of octahedron with three corners of each color, so there may be nothing in X that is stabilized by certain rotations. For the six ninety degree rotations about a corner we get $|X^g| = 0$. For the three 180 degree rotations about two opposite corners we get $|X^g| = 4$. For the eight 120 degree rotations about opposite faces we have $|X^g| = 2$. For each of the six 180 degree rotations about opposite edges we get $|X^g| = 0$. The identity has $|X^g| = C(6,3) = 20$. This gives us $\frac{1}{|G|}(3 \times 4 + 8 times 2 + 20) = \frac{48}{24} = 2$. Thus there are two distinct colorings of this type.]

Figure 3.64: Octahedra with Three Corners Each of Two Colors



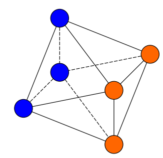
62. Draw these octahedra.
[Answer: See figure 3.64.]

63. How many distinct ways can we color the corners of an octahedron with two colors so that no opposite corners have the same color?
[Answer: Our group G is S_4. If a two color octahedron were fixed under any rotation about a side or edge, then we would need the colors of opposite corners to be the same. Thus $|X^g| = 0$ for those elements. For 120 degree rotations about a face we get $|X^g| = 2$. For the identity map we get $|X^g| = 2^3$ as three corners determine the other three. This gives us $\frac{1}{|G|}(8 \times 2 + 2^3) = \frac{24}{24} = 1$. Thus there is only one coloring of this type.]

64. Draw this octahedra.
[Answer: See figure 3.65.]

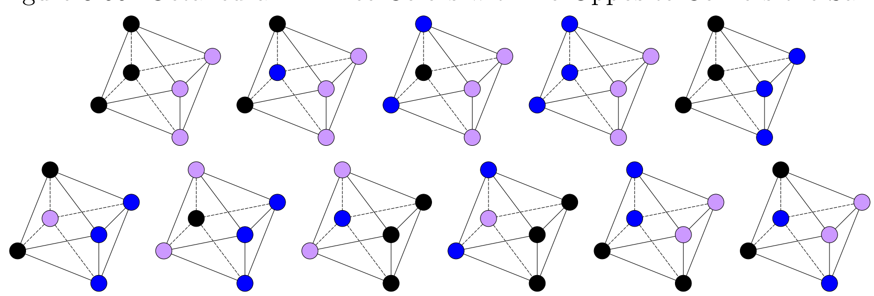
65. How many distinct ways can we color the corners of an octahedron with three colors so that no opposite corners have the same color?

Figure 3.65: Octahedra in Two Colors with No Opposite Corners the Same



[Answer: Our group G is S_4. We again get $|X^g| = 0$ for rotations about a side or edge. For the 120 degree rotations about a face we get $|X^g| = 3 \times 2$. For the identity map we get $|X^g| = 3^3 \times 2^3$ as three corners limit the choices for other three. This gives us $\frac{1}{|G|}(8 \times 6 + 216) = \frac{264}{24} = 11$. Thus there are eleven possible colorings.]

Figure 3.66: Octahedra in Three Colors with No Opposite Corners the Same

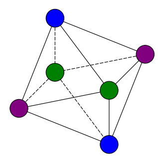


66. Draw these octahedra.
[Answer: See figure 3.66.]

67. How many distinct ways can we color the corners of an octahedron with n colors so that no opposite corners have the same color?
[Answer: Our group G is S_4. We again get $|X^g| = 0$ for rotations about a corner or edge. For the eight 120 rotations about a face we get $|X^g| = n \times (n-1)$. For the identity map we get $|X^g| = n^3 \times (n-1)^3$ as again three corners limit the choices for other three. This gives us $\frac{1}{|G|}(8 \times n \times (n-1) + n^3 \times (n-1)^3) = \frac{1}{24}(n-1)n(n^4 - 2n^3 + n^2 + 1)$.]

68. How many distinct ways can we color the corners of an octahedron with three colors so that no adjacent corners have the same color?
[Answer: Our group G is S_4. If a two color octahedron were fixed under any rotation about a face, an edge, or 90 degrees about a corner, then we would have adjacent corners of the same color. Thus $|X^g| = 0$ for those elements. For 180 degree rotations about a corner we get $|X^g| = 3! = 6$. The most involved part of this calculation comes from the identity element. Here we can use the fact that a repeated color must occur at opposite corners. Thus with only three colors we have every pair of opposite corners the same. This gives us $3! = 6$ possibilities for $|X^g| = 6$. We get $\frac{1}{|G|}(3 \times 6 + 6) = \frac{24}{24} = 1$. Thus there is only one coloring of this type.]

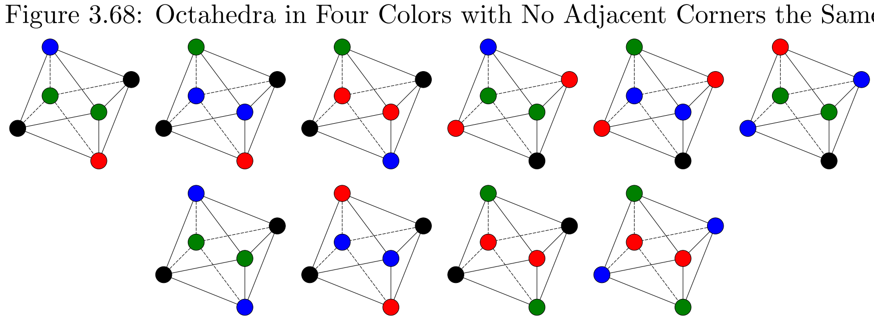
Figure 3.67: Octahedra in Three Colors with No Adjacent Corners the Same



69. Draw this octahedra.
    [Answer: See figure 3.67.]

70. How many distinct ways can we color the corners of an octahedron with four colors so that no adjacent corners have the same color?
    [Answer: Our group G is S_4. Again we get $|X^g| = 0$ for all group elements except 180 rotation about a corner and the identity. For our three rotations we get $|X^g| = 4 \times 3 \times 2 \times 2 = 48$. For the identity element we can break things down into the case where we use three of four colors, and the case where we use all four. The three color case has a total of $4! = 24$ possibilities. The all four color case requires exactly two opposite corner pairs to be equal. Here we get $3 \times 4 \times 3 \times 2 \times 1 = 72$. We first must pick which opposite pairs are equal in one of three ways, and then must color those leaving 2 ways to place the last two colors. Thus for $g = e$ we have $|X^g| = 24 + 72 = 96$. We thus get $\frac{1}{24}(3 \times 48 + 96) = 10$. Thus there are ten possible colorings of this type.]

Figure 3.68: Octahedra in Four Colors with No Adjacent Corners the Same

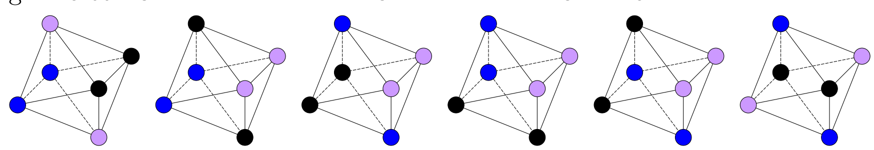


71. Draw these octahedra.
    [Answer: See figure 3.68.]

72. How many distinct ways can we color the corners of an octahedron with n colors so that no adjacent corners have the same color?
    [Answer: Our group G is S_4. Again we get $|X^g| = 0$ for all group elements except 180 rotation about a corner and the identity. For our three rotations we get $|X^g| = n(n-1)(n-2)^2$. For the identity element we can break things down into four separate cases. If we have exactly three repeated colors we get $n(n-1)(n-2)$ colorings. If we have exactly two we get $3n(n-1)(n-2)(n-3)$. For exactly three we get $3n(n-1)(n-2)(n-3)(n-4)$ and for no repeated colors we get $n(n-1)(n-2)(n-3)(n-4)(n-5)$ colorings. $\frac{1}{24}(3 \times n(n-1)(n-2)^2 + n(n-1)(n-2) + 3n(n-1)(n-2)(n-3) + 3n(n-1)(n-2)(n-3)(n-4) + n(n-1)(n-2)(n-3)(n-4)(n-5)) = \frac{1}{24}n(n-1)(n-2)(n^3 - 9n^2 + 32n - 38)$.

73. How many distinct ways can we color the corners of an octahedron with three colors so that no color appears more than twice?
[Answer: Our group G is S_4. Here any octahedron invariant under a ninety degree corner rotation or a 120 degree face rotation would have more than two sides of the same color. Thus we get $|X^g| = 0$ for these group elements. For the three 180 degree rotations about a corner we get $|X^g| = 3! = 6$. For the six 180 degree rotations through the midpoint of opposite edges we also get $|X^g| = 3! = 6$. For our count for the identity we know we must use each color twice. We have to pick two corners for the first color in fifteen ways, and then 2 remaining for the next in six ways. Then the last two are determined. Thus we get $|X^g| = 90$. For our final answer we compute $\frac{1}{24}(9 \times 6 + 90) = 6$ to conclude there are six distinct colorings.]

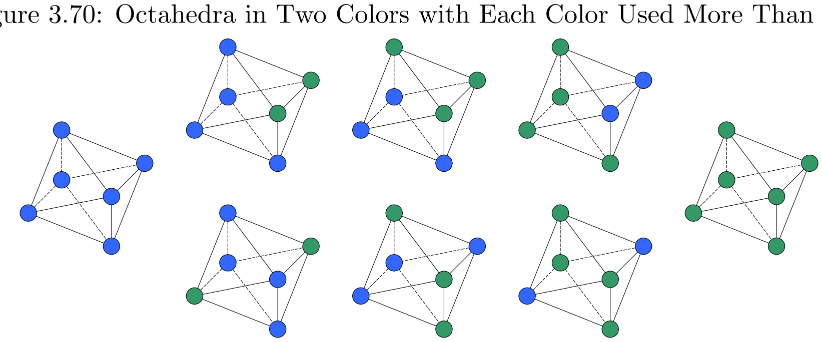
Figure 3.69: Octahedra in Three Colors with No Color Used More Than Twice



74. Draw these octahedra.
[Answer: See figure 3.69.]

75. How many distinct ways can we color the corners of an octahedron with two colors so that each color appears more than once?
[Answer: Our group G is S_4. Our set X is the set of two colored octahedra with each color appearing at least twice. For the eight 120 degree rotations about a corner we get $|X^g| = 2^2 = 4$. For the six 180 degree rotations about the midpoints of opposite edges we get $|X^g| = 2^3 = 8$. For the three 180 degree rotations about opposite faces we can break things down into the cases that the opposite faces have the same color, and they they have different colors. This gives us $|X^g| = 8 + 4 = 12$. For the six 90 degree face rotations we get $|X^g| = 2^2 = 4$. For the identity we can break things down into the three possibilities for a split. We either get four of one color and two of another, three of each, or all the same color. This gives us $|X^g| = 15 \times 2 + 10 \times 2 + 2$. Putting this together and using the lemma gives us $\frac{1}{24}(8 \times 4 + 6 \times 8 + 3 \times 12 + 6 \times 4 + 52) = \frac{192}{24} = 8$. Thus there are eight such colorings.]

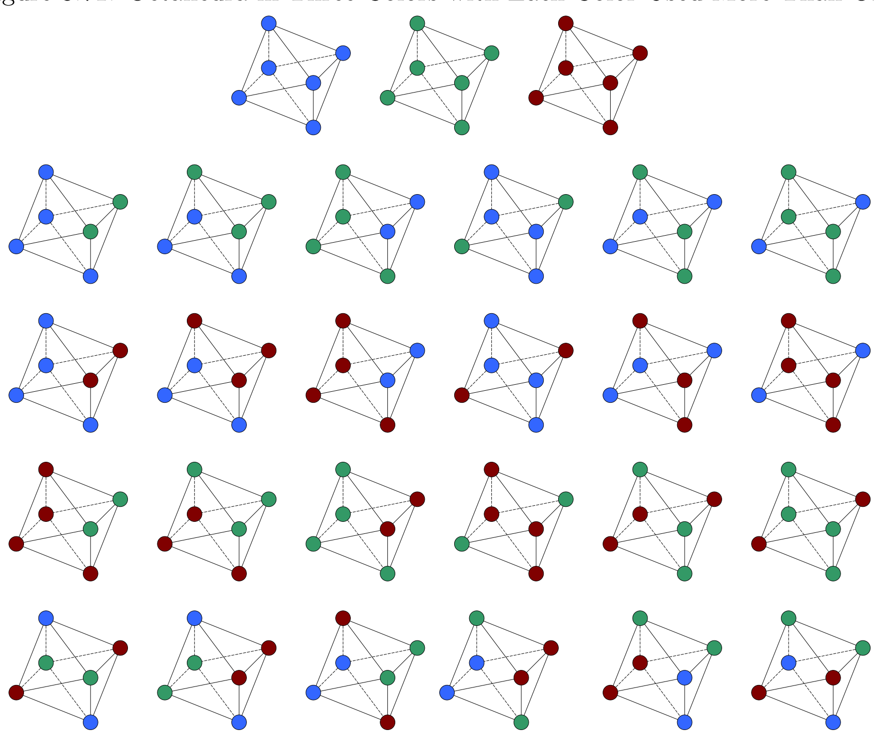
Figure 3.70: Octahedra in Two Colors with Each Color Used More Than Once



76. Draw these octahedra.
[Answer: See figure 3.71.]

77. How many distinct ways can we color the corners of an octahedron with three colors so that each color appears more than once?
[Answer: Our group G is S_4. Our set X is the set of three corner colored octahedra with each color appearing at least twice. For the eight 120 degree rotations about a corner we get $|X^g| = 3^2$. For the six 180 degree rotations about the midpoints of opposite edges we get $|X^g| = 3^3$. For the three 180 degree rotations about opposite faces we get $|X^g| = 3^3 + 2 \times 3 \times 2$. For the six 90 degree face rotations we get $|X^g| = 3^2$. For the identity we can break things down into the four possibilities for a split. We either get two each of three colors, four of one color and two of another, three of each, or all the same color. This gives us $|X^g| = 15 \times 3 \times 2 + 15 \times 3 \times 2 + 10 \times 3 \times 2 + 3 = 243$. Putting this together and using the lemma gives us $\frac{1}{24}(8 \times 3^2 + 6 \times 3^3 + 3 \times 39 + 6 \times 3^2 + 243) = 27$. Therefore there are twenty-seven such colorings.]

Figure 3.71: Octahedra in Three Colors with Each Color Used More Than Once



78. Draw these octahedra.
[Answer: See figure ??.]

79. How many distinct ways can we color the corners of an octahedron with n colors so that each color appears more than once?
[Answer: Our group G is S_4. Our set X is the set of n colored octahedra with each color appearing at

least twice. For the eight 120 degree rotations about a corner we get $|X^g| = n^2$. For the six 180 degree rotations about the midpoints of opposite edges we get $|X^g| = n^3$. For the three 180 degree rotations about opposite faces we can break things down into the cases that the opposite faces have the same color, and they they have different colors. This gives us $|X^g| = n^3 + 2n(n-1)$. For the six 90 degree face rotations we get $|X^g| = n^2$. For the identity we can break things down into the four possibilities for a split. We either get two each of three colors, four of one color and two of another, three of each, or all the same color. This gives us $|X^g| = 15n(n-1)(n-2) + 15n(n-1) + 10n(n-1) + n$. Putting this together and using the lemma gives us $\frac{1}{24}(8 \times n^2 + 6 \times n^3 + 3 \times (n^3 + 2n(n-1)) + 6 \times n^2 + 15n(n-1)(n-2) + 15n(n-1) + 10n(n-1) + n)$ which miraculously collapses down to n^3. This is sequence A000578 in the OEIS.]

80. How many distinct ways can we color the corners of an dodecahedron with two colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 2^4$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 2^8$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 2^{10}$. The identity has $|X^g| = 2^{20}$. This gives us $\frac{1}{|G|}(24 \times 2^4 + 20 \times 2^8 + 15 \times 2^10 + 2^{20}) = \frac{1069440}{60} = 17824$. Thus there are 17824 dodecahedra.]

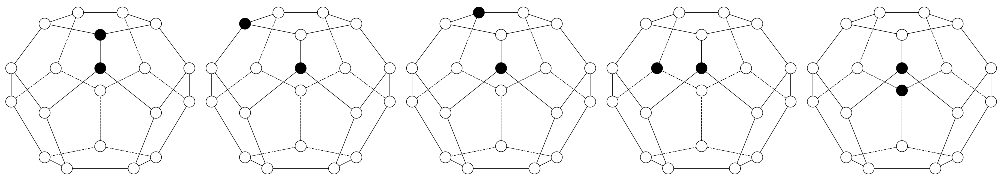
81. How many distinct ways can we color the corners of an dodecahedron with three colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 3^4$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 3^8$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 3^{10}$. The identity has $|X^g| = 3^{20}$. This gives us $\frac{1}{|G|}(24 \times 3^4 + 20 \times 3^8 + 15 \times 3^{10} + 3^{20}) = \frac{3487803300}{60} = 58130055$. Thus there are 58130055 dodecahedra.]

82. How many distinct ways can we color the corners of an dodecahedron with n colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = n^4$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = n^8$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = n^{10}$. The identity has $|X^g| = n^{20}$. This gives us $\frac{1}{|G|}(24n^4 + 20n^8 + 15n^{10} + n^{20}) = \frac{n^4}{60}(24 + 20n^4 + 15n^6 + n^{16})$.]

83. How many distinct ways can we color the corners of an dodecahedron with two colors so that exactly two are black and the rest are white?
[Answer: Our group G is A_5, but our set X is now the dodecahedron that have exactly two corners colored. This will drastically change each of our X^g, as most of the elements that were fixed before no longer exist in our new set. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 0$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 1$. This is because only the opposite corners we are rotating about can be colored black for the coloring to be fixed. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 10$. The identity has $|X^g| = C(20, 2) = 190$. This gives us $\frac{1}{|G|}(20 \times 1 + 15 \times 10 + 190) = \frac{360}{60} = 6$. Thus there are six distinct dodecahedra.]

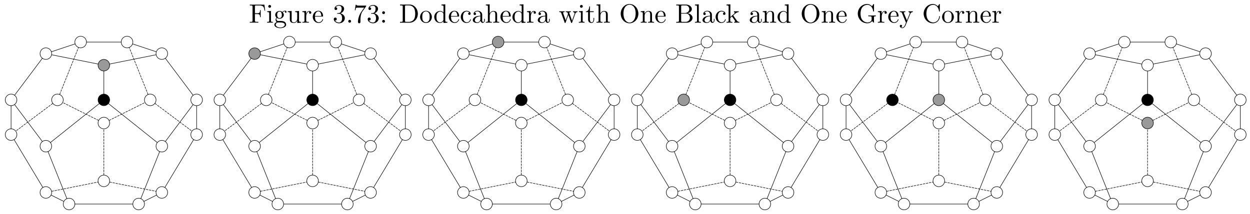
84. Draw these dodecahedra.
[Answer: See figure 3.72.]

Figure 3.72: Dodecahedra with Exactly Two Black Corners



85. How many distinct ways can we color the corners of an dodecahedron with two colors so that one is grey, one is black and all the rest are white?
[Answer: Our group G is A_5, but our set X is now the dodecahedron that have exactly two corners colored in different colors. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 0$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 2$. For the fifteen 180 degree rotations about the midpoints of opposite edges we now have $|X^g| = 0$. The identity has $|X^g| = C(20, 1)C(19, 1) = 380$. This gives us $\frac{1}{|G|}(20 \times 2 + 380) = \frac{420}{60} = 7$. Thus there are seven distinct dodecahedra with these colorings.]

Figure 3.73: Dodecahedra with One Black and One Grey Corner



86. Draw these dodecahedra.
[Answer: See figure 3.73.]

87. How many distinct ways can we color the corners of an dodecahedron with two colors so that exactly three are black and the rest are white?
[Answer: Our group G is A_5, and our set X is the dodecahedron that have exactly three corners colored. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 0$. For the $2 \times 10 = 20$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 6$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 0$. The identity has $|X^g| = C(20, 3) = 190$. This gives us $\frac{1}{|G|}(20 \times 6 + 1140) = \frac{1260}{60} = 21$. Thus there are twenty-one distinct dodecahedra.]

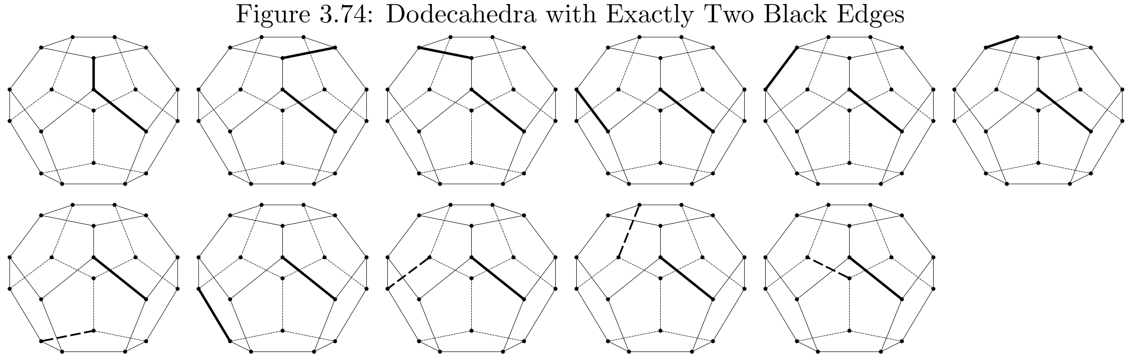
88. How many distinct ways can we color the edges of an dodecahedron with two colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 2^6$. For the $2 \times 10 = 40$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 2^{10}$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 2^{16}$. The identity has $|X^g| = 2^{30}$. This gives us $\frac{1}{|G|}(24 \times 2^6 + 20 \times 2^{10} + 15 \times 2^{16} + 2^{30}) = \frac{1074746880}{60} = 17912448$. Thus there are 17912448 dodecahedra.]

89. How many distinct ways can we color the edges of an dodecahedron with three colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 3^6$. For the $2 \times 10 = 40$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 3^{10}$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 3^{16}$. The identity has $|X^g| = 3^{30}$. This gives us $\frac{1}{|G|}(24 \times 3^6 + 20 \times 3^{10} + 15 \times 3^{16} + 3^{30}) = \frac{205891778993940}{60} = 3431529649899$. Thus there are 3431529649899 dodecahedra.]

90. How many distinct ways can we color the edges of an dodecahedron with n colors?
[Answer: Our group G is A_5. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = n^6$. For the $2 \times 10 = 40$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = n^{10}$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = n^{16}$. The identity has $|X^g| = n^{30}$. This gives us $\frac{1}{|G|}(24n^6 + 20n^{10} + 15n^{16} + n^{30}) = \frac{n^6}{60}(24 + 20n^4 + 15n^{10} + n^{24})$.]

91. How many distinct ways can we color the edges of an dodecahedron with two colors so that exactly two edges are black and the rest are white?
[Answer: Our group G is A_5. Our set X is the collection of dodecahedra with exactly two black edges. For the $4 \times 6 = 24$ nontrivial rotations about a face of multiples of 72 degrees we get $|X^g| = 0$. For the $2 \times 10 = 40$ 120 degree rotations about two opposite corners in either direction we get $|X^g| = 0$. For the fifteen 180 degree rotations about the midpoints of opposite edges we have $|X^g| = 15$. The identity has $|X^g| = C(30, 2) = 435$. This gives us $\frac{1}{|G|}(15 \times 15 + 435) = \frac{660}{60} = 11$. Thus there are 11 such dodecahedra.]

Figure 3.74: Dodecahedra with Exactly Two Black Edges



92. Draw these dodecahedra.
[Answer: See figure 3.74.]

3.4 Matrices

For the following problems let $M_{m\times n}(R)$ be the set of matrices with entries in R.

1. How many distinct elements in $M_{2\times 2}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other? For example, here we would consider $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ to be the same.

[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{2\times 2}(\mathbb{Z}_3)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = 2^3$ and for e we get $|X^g| = 2^4$. Thus we have $\frac{1}{|G|}(1 \times 2^3 + 1 \times 2^4) = \frac{1}{2}(8 + 16) = 12$. Thus there are twelve possibilities.]

2. List each of these possibilities.
[Answer: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.]

3. How many distinct elements in $M_{2\times 2}(\mathbb{Z}_3)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other?
[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{2\times 2}(\mathbb{Z}_3)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = 3^3$ and for e we get $|X^g| = 3^4$. Thus we have $\frac{1}{|G|}(1 \times 3^3 + 1 \times 3^4) = \frac{1}{2}(27 + 81) = 54$. Thus there are fifty-four elements equivalent up to taking the transpose.]

4. How many distinct elements in $M_{2\times 2}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other?
[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{2\times 2}(\mathbb{Z}_n)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = n^3$ and for e we get $|X^g| = n^4$. Thus we get $\frac{1}{|G|}(1 \times n^3 + 1 \times n^4) = \frac{n^3}{2}(1 + n)$.]

5. How many distinct elements in $M_{3\times 3}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other? For example, here we would consider $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to be the same.
[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{3\times 3}(\mathbb{Z}_3)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = 2^6$ and for e we get $|X^g| = 2^9$. Thus we have $\frac{1}{|G|}(1 \times 2^6 + 1 \times 2^9) = \frac{1}{2}(576) = 288$. Thus there are 288 possibilities.]

6. How many distinct elements in $M_{3\times 3}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other?
[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{3\times 3}(\mathbb{Z}_n)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = n^6$ and for e we get $|X^g| = n^9$. Thus we get $\frac{1}{|G|}(1 \times n^6 + 1 \times n^9) = \frac{n^6}{2}(1 + n^3)$.]

7. How many distinct elements in $M_{m\times m}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent if one is the transpose of the other?
[Answer: Let $G = \{e, t\}$ which is isomorphic to $\mathbb{Z}_2$. $X = M_{m\times m}(\mathbb{Z}_n)$. Consider the action where e fixes all of X and $t \cdot A = A^T$ for any A in X. For t we get $|X^g| = n^{m(m+1)/2}$ and for e we get $|X^g| = n^{m^2}$. Thus we get $\frac{1}{2}(n^{m(m+1)/2} + n^{m^2})$.]

8. How many elements in $M_{2\times 2}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees? For example, here we would consider

$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ to be the same.
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 2$, for r^2 we have $|X^g| = 2^2$ and for e we get $|X^g| = 2^4$. Thus we have $\frac{1}{|G|}(2 \times 2^1 + 1 \times 2^2 + 1 \times 2^4) = \frac{1}{4}(4 + 4 + 16) = 6$. This gives us six possibilities.]

9. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$.]

10. How many elements in $M_{2\times 2}(\mathbb{Z}_3)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees? For example, here we would consider $\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}$, $\begin{bmatrix}0 & 0\\ 2 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & 0\\ 1 & 0\end{bmatrix}$ to be the same.
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 3^1$, for r^2 we have $|X^g| = 3^2$ and for e we get $|X^g| = 3^4$. Thus we have $\frac{1}{|G|}(2 \times 3^1 + 1 \times 3^2 + 1 \times 3^4) = \frac{1}{4}(96) = 24$. This gives us twenty-four possibilities.]

11. How many elements in $M_{2\times 2}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees?
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = n^1$, for r^2 we have $|X^g| = n^2$ and for e we get $|X^g| = n^4$. Thus we get $\frac{1}{|G|}(2 \times n^1 + 1 \times n^2 + 1 \times n^4) = \frac{n}{4}(n+1)(n^2 - n + 2)$.]

12. How many elements in $M_{3\times 3}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees?
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 2^3$, for r^2 we have $|X^g| = 2^5$ and for e we get $|X^g| = 2^9$. Thus we have $\frac{1}{|G|}(2 \times 2^3 + 1 \times 2^5 + 1 \times 2^9) = \frac{1}{4}(560) = 140$. This gives us 140 possibilities.]

13. How many elements in $M_{4\times 4}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees?
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 2^4$, for r^2 we have $|X^g| = 2^8$ and for e we get $|X^g| = 2^{16}$. Thus we have $\frac{1}{|G|}(2 \times 2^4 + 1 \times 2^8 + 1 \times 2^{16}) = \frac{65824}{4} = 16456$. This gives us 16456 possibilities.]

14. How many elements in $M_{5\times 5}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees?
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 2^7$, for r^2 we have $|X^g| = 2^{13}$ and for e we get $|X^g| = 2^{25}$. Thus we have $\frac{1}{|G|}(2 \times 2^7 + 1 \times 2^{13} + 1 \times 2^{25}) = \frac{33562880}{4} = 8390720$. This gives us 16456 possibilities.]

15. How many elements in $M_{2k\times 2k}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be rotated to get the other by some multiple of ninety degrees?
[Answer: Let $G = \{e, r, r^2, r^3\}$ be the cyclic group of order four, and let r rotate any matrix clockwise by ninety degrees. For r and r^3 we get $|X^g| = 2^{k^2}$, for r^2 we have $|X^g| = 2^{2k^2}$ and for e we get $|X^g| = 2^{4k^2}$. Thus we have $\frac{1}{|G|}(2 \times 2^{k^2} + 1 \times 2^{2k^2} + 1 \times 2^{4k^2}) = \frac{1}{4}(2^{k^2+1} + 2^{2k^2} + 2^{4k^2})$.]

16. Consider two elements of $M_{2\times 2}(\mathbb{Z}_2)$ to be equivalent if you can get from one to the other by permuting the rows. How many representatives are there?
[Answer: We have an S_2 action here with $|X^g| = 2^2$ if $g = (1,2)$ and $|X^g| = 2^4$ if $g = e$. We get $\frac{1}{2}(2^2 + 2^4) = 10$.]

17. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$.]

18. Consider two elements of $M_{3\times 3}(\mathbb{Z}_2)$ to be equivalent if you can get from one to the other by permuting the rows. How many representatives are there?
[Answer: We have an S_2 action here with $|X^g| = 2^6$ for 2-cycles, $|X^g| = 2^3$ for 3-cycles and $|X^g| = 2^9$ for $g = e$. We get $\frac{1}{6}(3 \times 2^6 + 2 \times 2^3 + 2^9) = 120$.]

19. Consider two elements of $M_{2\times 2}(\mathbb{Z}_2)$ to be equivalent if you can get from one to the other by permuting the rows and/or columns. How many representatives are there?
[Answer: We actually have an $S_2 \times S_2$ action here as row permutations and column permutations commute with each other. We get $|X^g| = 2^2$ for everything except the identity, which has $|X^g| = 2^4$.. We get $\frac{1}{4}(3 \times 2^2 + 2^4) = 7$.]

20. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$.]

21. How many elements in $M_{2\times 2}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be flipped to get the other by some combination of vertical and horizontal reflections?
[Answer: A vertical and horizontal flip make a rotation by 180 degrees, so we are really just considering an action of D_2. Note that if f is a flip about a vertical axis and r is 180 degree rotation then rf becomes the flip about a horizontal one. Let $G = \{e, r, f, rf\}$ be the dihedral group D_2 and allow this to act on our matrices by rotations and reflections. For r, f and rf we get $|X^g| = 2^2$ and for e we get $|X^g| = 2^4$. Thus we have $\frac{1}{|G|}(3 \times 2^2 + 1 \times 2^4) = \frac{28}{4} = 7$. This gives us seven possibilities.]

22. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$.]

23. How many elements in $M_{2\times 2}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent if one can be flipped to get the other by some combination of vertical and horizontal reflections?
[Answer: Let $G = \{e, r, f, rf\}$ be the dihedral group D_2 and allow this to act on our matrices by rotations and reflections. For r, f and rf we get $|X^g| = n^2$ and for e we get $|X^g| = n^4$. Thus we get $\frac{1}{|G|}(3 \times n^2 + 1 \times n^4) = \frac{n^2(3+n^2)}{4}$.]

24. How many elements in $M_{3\times 3}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent if one can be flipped to get the other by some combination of vertical and horizontal reflections?
[Answer: Let $G = \{e, r, f, rf\}$ be the dihedral group D_2 and allow this to act on our matrices by rotations and reflections. For f and rf we get $|X^g| = 2^6$. For r we only get 2^3. Finally, for e we get $|X^g| = 2^9$. Thus we have $\frac{1}{|G|}(2 \times 2^6 + 2^3 + 1 \times 2^4) = \frac{152}{4} = 38$. This gives us thirty-eight possibilities.]

25. How many elements in $M_{2\times 2}(\mathbb{Z}_2)$ do we get if we consider two matrices to be equivalent under the action of D_4 with the usual rotations and reflections?
[Answer:Let $G = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$ be the dihedral group D_4 and allow this to act on our matrices by rotations and reflections. For r and r^3 we get $|X^g| = 2^1$. For r^2 we get $|X^g| = 2^2$. For the two diagonal reflections rf and r^3f we get $|X^g| = 2^3$. For f and r^2f we get $|X^g| = 2^2$. Finally, for e we get $|X^g| = 2^4$. Thus we have $\frac{1}{|G|}(2\times 2^1 + 1\times 2^2 + 2\times 2^3 + 2\times 2^2 + 2^4) = \frac{48}{8} = 6$. This gives us six possibilities.]

26. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$.]

27. How many elements in $M_{2\times 2}(\mathbb{Z}_4)$ do we get if we consider two matrices to be equivalent under the action of D_4 with the usual rotations and reflections?
[Answer:Let $G = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$ be the dihedral group D_4 and allow this to act on our matrices by rotations and reflections. For r and r^3 we get $|X^g| = 4^1$. For r^2 we get $|X^g| = 4^2$. For the two diagonal reflections rf and r^3f we get $|X^g| = 4^3$. For f and r^2f we get $|X^g| = 4^2$. Finally, for e we get $|X^g| = 4^4$. Thus we have $\frac{1}{|G|}(2\times 4^1 + 1\times 4^2 + 2\times 4^3 + 2\times 4^2 + 4^4) = \frac{440}{8} = 55$. This gives us fifty-five possibilities.]

28. How many elements in $M_{2\times 2}(\mathbb{Z}_n)$ do we get if we consider two matrices to be equivalent under the action of D_4 with the usual rotations and reflections?
[Answer:Let $G = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$ be the dihedral group D_4 and allow this to act on our matrices by rotations and reflections. For r and r^3 we get $|X^g| = n^1$. For r^2 we get $|X^g| = n^2$. For the two diagonal reflections rf and r^3f we get $|X^g| = n^3$. For f and r^2f we get $|X^g| = n^2$. Finally, for e we get $|X^g| = n^4$. Thus we have $\frac{1}{|G|}(2\times n^1 + 1\times n^2 + 2\times n^3 + 2\times n^2 + n^4) = \frac{n(n+1)(n^2+n+2)}{8}$.]

29. Let X be the elements of the group $GL(2,2)$. This is the set of all invertible matrices in $M_{2\times 2}(\mathbb{Z}_2)$. Specifically, it contains $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$, and $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$. Consider two elements to be equivalent if you can get from one to the other by permuting the rows. How many representatives are there?
[Answer: We have an S_2 action here, and $|X^g| = 0$ for the non-identity element. For the identity $|X^g| = 6$ giving us $\frac{1}{2}(6) = 2$.]

30. List each of these possibilities.
[Answer: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.]

31. Let X be the elements of the group $GL(2,2)$. This is the set of all invertible matrices in $M_{2\times 2}(\mathbb{Z}_2)$. Consider two elements to be equivalent if you can get from one to the other by rotation of some multiple of ninety degrees. How many representatives are there?
[Answer: We have an $\mathbb{Z}_4$ action, and can think of our group as $\{e, r, r^2, r^3\}$. We get $|X^g| = 0$ for r and r^3. We get $|X^g| = 2$ for r^2 and $|X^g| = 6$ for the identity. Thus we have $\frac{1}{4}(2+6) = 2$.]

32. List each of these possibilities.
[Answer: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$.]

33. Let X be the elements of the group $GL(2,2)$. This is the set of all invertible matrices in $M_{2\times 2}(\mathbb{Z}_2)$. Consider two elements to be equivalent if you can get from one to the other by permuting the rows and/or columns. How many representatives are there?
[Answer: We have an $S_2 \times S_2$ action here as row permutations and column permutations commute with each other. We could also see this as an action of V or $\mathbb{Z}_2$. We get $|X^g| = 0$ for the elements that switch two rows and switch two columns. We get $|X^g| = 2$ for the element switching both and $|X^g| = 6$ for the identity. Thus we have $\frac{1}{4}(2+6) = 2$.]

34. List each of these possibilities.
[Answer: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$.]

35. Let X be the elements of the group $GL(2,2)$. This is the set of all invertible matrices in $M_{2\times 2}(\mathbb{Z}_2)$. Consider two elements to be equivalent if you can get from one to the other by inversion. How many representatives are there?
[Answer: We have an S_2 action here, as taking the inverse twice gives the original matrix. We get $|X^g| = 4$ for the element that performs the inverse, and $|X^g| = 6$ for the identity. Thus we have $\frac{4+6}{2} = 5$.]

36. List each of these possibilities.
[Answer: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$.]

37. Let X be the elements of the group $GL(2,3)$. This is the set of all invertible matrices in $M_{2\times 2}(\mathbb{Z}_3)$. This set contains forty-eight matrices. To see this note that there are 3^2-1 possibilities for the top row, and one that is chosen, the bottom row must not be a scalar multiple of that row since the two must be linearly independent. Thus there are 3^2-3 possibilities for the second row. We get $(3^2-1)(3^2-3) = 48$ possibilities in total. Consider two elements to be equivalent if you can get from one to the other by permuting the rows. How many representatives are there?
[Answer: We have an S_2 action here. If a matrix were fixed under switching rows, then the rows would be the same, implying the determinant is zero. Thus, here $|X^g| = 0$ for that element and $|X^g| = 48$ for the identity. Thus we have $\frac{48}{2} = 24$ representatives.]

38. List each of these possibilities.
[Answer: $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}, \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 2\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 2\end{bmatrix}, \begin{bmatrix}0 & 2\\2 & 1\end{bmatrix},$
$\begin{bmatrix}0 & 2\\2 & 2\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\2 & 2\end{bmatrix}, \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\1 & 2\end{bmatrix}, \begin{bmatrix}2 & 0\\2 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\2 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}2 & 2\\1 & 2\end{bmatrix},$
$\begin{bmatrix}2 & 2\\2 & 1\end{bmatrix}$.]

39. Note that switching the rows of a two by two matrix multiplies the determinant by negative one. Therefore, we should be able to pick a representative from each pair with determinant one when listing matrices for the last problem. Repeat the problem in such a way that each of our representatives has determinant one. Note that these are now the elements of $SL(2,3)$, the collection of matrices of determinant one, and they form their own group under multiplication.
[Answer: $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 2\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 2\\1 & 2\end{bmatrix}, \begin{bmatrix}2 & 1\\0 & 2\end{bmatrix},$

$\begin{bmatrix}2&2\\0&2\end{bmatrix}$, $\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $\begin{bmatrix}1&2\\1&0\end{bmatrix}$, $\begin{bmatrix}1&0\\2&1\end{bmatrix}$, $\begin{bmatrix}2&2\\1&0\end{bmatrix}$, $\begin{bmatrix}1&1\\2&0\end{bmatrix}$, $\begin{bmatrix}2&0\\1&2\end{bmatrix}$, $\begin{bmatrix}2&1\\2&0\end{bmatrix}$, $\begin{bmatrix}2&0\\2&2\end{bmatrix}$, $\begin{bmatrix}1&1\\1&2\end{bmatrix}$, $\begin{bmatrix}2&1\\1&1\end{bmatrix}$, $\begin{bmatrix}1&2\\2&2\end{bmatrix}$, $\begin{bmatrix}2&2\\2&1\end{bmatrix}$.]

40. Consider the twenty-four elements of the group $SL(2,3)$. If we multiply all entries by two, this is the equivalent of multiplying by the matrix $\begin{bmatrix}2&0\\0&2\end{bmatrix}$, which has determinant one. Therefore $SL(2,3)$ is closed under this operation. If we consider two matrices here to be the same if they differ by a multiple of two, how many possibilities do we get?
[Answer: Multiplying by $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ twice is the same as multiplying by the identity. Thus we have an S_2 action. Multiplying by two leaves no matrix here unchanged and thus the action of that operation has $|X^g| = 0$ but the action of the identity has $|X^g| = 24$. This gives us $\frac{24}{2} = 12$ representatives.]

41. List each of these possibilities. These are coset representatives for $PSL(2,3)$, the quotient group of $SL(2,3)$ by the group $\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2&0\\0&2\end{bmatrix}\}$. This means, among other things, that the product of any two of these is a multiple of exactly one other matrix in the collection.
[Answer: $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\2&0\end{bmatrix}$, $\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\2&1\end{bmatrix}$, $\begin{bmatrix}0&1\\2&2\end{bmatrix}$, $\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $\begin{bmatrix}1&2\\1&0\end{bmatrix}$, $\begin{bmatrix}1&0\\2&1\end{bmatrix}$, $\begin{bmatrix}2&2\\1&0\end{bmatrix}$, $\begin{bmatrix}1&1\\1&2\end{bmatrix}$, $\begin{bmatrix}2&1\\1&1\end{bmatrix}$.]

42. Consider an atypical D_4 action on $M_{2\times 2}(\mathbb{Z}_4)$. Let r add one (modulo four) to each entry of the matrix, and n negates each entry. Recall in $\mathbb{Z}_4$ that $-1 = 3, -2 = 2, -3 = 1$ and $-0 = 0$. Notice that $a^4 = e$, $n^2 = e$, and $na = a^3n$, which are precisely the relations of D_4. How many distinct matrices do we get under this equivalence?
[Answer: For a, a^2 and a^3 we get $|X^g| = 0$. For n we get $|X^g| = 2^4$, because in order to remain fixed, each entry must be even. Note that a^2n switches evens and fixes odds which also gives $|X^g| = 2^4$. Since negating and adding one or three never fixes a number in $\mathbb{Z}_4$ we get $|X^g| = 0$ Finally, for e we have $|X^g| = 4^4$. Thus we have $\frac{1}{|G|}(2 \times 2^4 + 1 \times 4^4) = \frac{288}{8} = 36$.]

43. List each of these possibilities.
[Answer: $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&0\end{bmatrix}$, $\begin{bmatrix}0&0\\2&0\end{bmatrix}$, $\begin{bmatrix}0&0\\0&1\end{bmatrix}$, $\begin{bmatrix}0&0\\1&1\end{bmatrix}$, $\begin{bmatrix}0&0\\2&1\end{bmatrix}$, $\begin{bmatrix}0&0\\3&1\end{bmatrix}$, $\begin{bmatrix}0&0\\0&2\end{bmatrix}$, $\begin{bmatrix}0&0\\1&2\end{bmatrix}$, $\begin{bmatrix}0&0\\2&2\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $\begin{bmatrix}0&1\\2&0\end{bmatrix}$, $\begin{bmatrix}0&1\\3&0\end{bmatrix}$, $\begin{bmatrix}0&1\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\1&1\end{bmatrix}$, $\begin{bmatrix}0&1\\2&1\end{bmatrix}$, $\begin{bmatrix}0&1\\3&1\end{bmatrix}$, $\begin{bmatrix}0&1\\0&2\end{bmatrix}$, $\begin{bmatrix}0&1\\1&2\end{bmatrix}$, $\begin{bmatrix}0&1\\2&2\end{bmatrix}$, $\begin{bmatrix}0&1\\3&2\end{bmatrix}$, $\begin{bmatrix}0&1\\0&3\end{bmatrix}$, $\begin{bmatrix}0&1\\1&3\end{bmatrix}$, $\begin{bmatrix}0&1\\2&3\end{bmatrix}$, $\begin{bmatrix}0&1\\3&3\end{bmatrix}$, $\begin{bmatrix}0&2\\0&0\end{bmatrix}$, $\begin{bmatrix}0&2\\1&0\end{bmatrix}$, $\begin{bmatrix}0&2\\2&0\end{bmatrix}$, $\begin{bmatrix}0&2\\0&1\end{bmatrix}$, $\begin{bmatrix}0&2\\1&1\end{bmatrix}$, $\begin{bmatrix}0&2\\2&1\end{bmatrix}$, $\begin{bmatrix}0&2\\3&1\end{bmatrix}$, $\begin{bmatrix}0&2\\0&2\end{bmatrix}$, $\begin{bmatrix}0&2\\1&2\end{bmatrix}$, $\begin{bmatrix}0&2\\2&2\end{bmatrix}$, .]

44. Now consider a $\mathbb{Z}_4 \times \mathbb{Z}_4$ action on $M_{2\times 2}(\mathbb{Z}_4)$. To make things clearer, describe G as $\{e_1, r, r^2, r^3\} \times \{e_2, a, a^2, a^3\}$ and let the pair (r^j, a^k) act on matrices by adding one k times and rotating j times. It does not matter which we do first as these operations commute. How many distinct matrices do we get under this equivalence?

[Answer: For (r, e_2) and (r^3, e_2) we get $|X^g| = 4^1$. For (r^2, e_2) we get $|X^g| = 4^2$. For $(e_1, a), (e_1, a^2)$ and (e_1, a^3) we still get $|X^g| = 0$. For (r, a),(r^3, a),(r, a^2),(r^3, a^2),(r, a^3) and (r^3, a^3) we get $|X^g| = 4^1$. For (r^2, a) and (r^2, a^3) we get $|X^g| = 0$. For (r^2, a^2) we get $|X^g| = 4^2$. Finally, for $e = (e_1, e_2)$ we have $|X^g| = 4^4$. Thus we have $\frac{1}{|G|}(2 \times 4^1 + 1 \times 4^2 + 6 \times 4^1 + 1 \times 4^2 + 1 \times 4^4) = \frac{320}{16} = 20$.]

45. List each of these possibilities.
[Answer: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, .]

46. Next consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $M_{2\times 2}(\mathbb{Z}_4)$ made from $\{e_1, f\}$ and $\{e_2, n\}$ from the previous questions here. To be specific f reflects a matrix about a vertical line of symmetry and n negates each entry. Note that $fn = nf$ so our product is direct. How many orbits do we have with this group action?
[Answer: For (e_1, n) we get $|X^g| = 2^4$ and for (f, e_2) we get $|X^g| = 4^2$. For (f, n) we have $|X^g| = 4^2$. Finally the identity gives us $|X^g| = 4^4$. We get $\frac{1}{|G|}(2^4 + 2 \times 4^2 + 4^4) = \frac{304}{4} = 76$.]

47. Next consider a $D_4 \times \mathbb{Z}_2$ action on $M_{2\times 2}(\mathbb{Z}_4)$ made from $\{e_1, r, r^2, r^3, f, rf, r^2f, r^3f\}$ and $\{e_2, n\}$ from the previous questions here. Specifically, r rotates a matrix ninety degrees clockwise, f flips it across a vertical line of symmetry, and n negates each entry. Note that $rn = nr$ and $fn = nf$ so our product is direct. Also recall that $r^4 = f^2 = e$, $fr = r^3f$, and $n^2 = e$, so we have one copy of D_4 in the first slot and a copy of $\mathbb{Z}_2$ in the second. How many orbits do we have with this group action?
[Answer: From previous questions or by recalculating, we get $|X^g| = 4$ for (r, e_2) and (r^3, e_2), $|X^g| = 4^2$ for (r^2, e_2), (f, e_2) and (r^2f, e_2), $|X^g| = 4^3$ for (rf, e_2) and (r^3f, e_2), $|X^g| = 2^4$ for (e_1, n), and $|X^g| = 4^4$ for (e_1, e_2). For (r, n) and (r^3, n) we get $|X^g| = 4^1$. For $(r^2, n), |X^g| = 4^2$. For (f, n) and (r^2f, n) we have $|X^g| = 4^2$ and for (f, n) and (r^2f, n) we have $|X^g| = 2^2 \cdot 4$. Putting this all together we get $\frac{1}{|G|}(4 \times 4^1 + 9 \times 4^2 + 2 \times 4^3 + 1 \times 4^4) = \frac{544}{16} = 34.$.]

48. Now consider a $\mathbb{Z}_2 \times D_4$ action on $M_{2\times 2}(\mathbb{Z}_4)$ made from $\{e_1, f\}$ and $\{e_2, a, a^2, a^3, n, an, a^2n, a^3n\}$ from the previous questions here. How many orbits do we have with this group action?
[Answer: Mostly we can take from previous calculations. We do need to compute elements fixed under $(f, e), (f, a), (f, a^2), (f, a^3), (f, n), (f, an), (f, a^2n)$, and (f, a^3n), each of which has $|X^g| = 4^2$ except for (f, a) and (f, a^3) which fix no elements at all. We end up with $\frac{1}{|G|}(8 \times 4^2 + 1 \times 4^4) = \frac{384}{16} = 24$.]

49. Next consider a $D_4 \times D_4$ action on $M_{2\times 2}(\mathbb{Z}_4)$ made from $\{e_1, r, r^2, r^3, f, rf, r^2f, r^3f\}$ and $\{e_2, a, a^2, a^3, n, an, a^2n, a^3n\}$ from the previous questions here. To be specific r rotates a matrix ninety degrees clockwise, f flips it across a vertical line of symmetry, a adds one to all entries, and n negates each entry. Note that $ra = ar, rn = nr, fa = af$, and $fn = nf$ so our product is direct. Also recall that $r^4 = f^2 = e$, $fr = r^3f$, $a^4 = n^2 = e$ and $na = a^3n$, so we have a copy of D_4 in each slot. How many orbits do we have with this group action?
[Answer: For a group of size sixty-four, it is best to make a table for $|X^g|$. We arrive at the following.

	e_2	a	a^2	a^3	n	an	a^2n	a^3n
e	4^4	0	0	0	2^4	0	2^4	0
r	4^1	4^1	4^1	4^1	4^1	4^1	4^1	4^1
r^2	4^2	0	4^2	0	4^2	4^2	4^2	4^2
r^3	4^1	4^1	4^1	4^1	4^1	4^1	4^1	4^1
f	4^2	0	4^2	0	4^2	4^2	4^2	4^2
rf	4^3	0	0	0	$2^2 \cdot 4$	0	$2^2 \cdot 4$	0
r^2f	4^2	0	4^2	0	4^2	4^2	4^2	4^2
r^3f	4^3	0	0	0	$2^2 \cdot 4^1$	0	$2^2 \cdot 4^1$	0

The exponents have been left in intentionally, to better explain the origins of the calculations involved. Now we simply add each of the terms in this table and divide by 64. This gives us $\frac{832}{64} = 13$.

50. List each of these possibilities.
[Answer: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$.]

3.5 The Proof of the Cauchy Frobenius Theorem

1. In this first set of exercises prove the Cauchy Frobenius theorem in full generality. For any action of a group G on the set X, this theorem gives a formula for $|X\backslash G|$, the number of distinct orbits. Specifically it says

$$|X\backslash G| = \frac{1}{|G|} \cdot \sum_{g \in G} |X^g|.$$

Our argument starts with the summation in our formula and uses the Orbit-Stabilizer Theorem to show this is equal to $|X\backslash G|$.

(a) Prove that $\sum_{g \in G} |X^g|$ is equal to $\sum_{x \in X} |stab(x)|$.
[Answer: We use the fact that they both are counting the same thing, which is the total number of x and g so that $gx = g$. We get $\sum_{g \in G} |X^g| = \sum_{g \in G} |\{x \in X : gx = x\}| = |\{(g, x) \in G \times X : gx = x\}| = \sum_{x \in X} |\{g \in G : gx = x\}| = \sum_{x \in X} |stab(x)|$.]

(b) Prove that $\sum_{x \in X} |stab(x)| = |G| \cdot |X\backslash G|$.
[Answer: By the Orbit-Stabilizer theorem we know $\sum_{x \in X} |stab(x)| = \sum_{x \in X} \frac{|G|}{|orb(x)|}$. Notice that the $|G|$ term is constant because the size of the group is most certainly not changing. Therefore we can rewrite this as $|G| \cdot \sum_{x \in X} \frac{1}{|orb(x)|}$. We are summing over every x in X, so we might as well sum over each orbit one at a time. Thus instead of summing just over X, we take each orbit A in $X\backslash G$ and then each x in A. Since our orbits form a partition of X, we still count every element in X and count no element twice. We have $|G| \cdot \sum_{A \in X\backslash G} \sum_{x \in A} \frac{1}{|orb(x)|} = |G| \cdot \sum_{A \in X\backslash G} \frac{1}{1} = |G| \cdot |X\backslash G|$.]

2. In this set of exercises we walk through the first part proof with a specific example. Let D_4 act on the set of all invertible two-by-two matrices with entries in $\mathbb{Z}_2$. Assume that f reflects a matrix about a vertical axis of symmetry and r rotates a matrix by ninety degrees.

   (a) Find $|X^g|$ for each $g \in G$.
   [Answer: $X^e = X$, $X^r = X^{r^3} = X^f = X^{r^2 f} = \emptyset$, $X^{r^2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$,
   $X^{rf} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$, $X^{r^3 rf} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.]

   (b) For each $g \in G$, list $\{x \in X : gx = x\}$.
   [Answer:

   - $g = e$: X
   - $g = r$: $\emptyset$
   - $g = r^2$: $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
   - $g = r^3$: $\emptyset$
   - $g = f$: $\emptyset$
   - $g = rf$: $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$
   - $g = r^2 f$: $\emptyset$
   - $g = r^3 f$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \right\}$]

   (c) Find $\sum_{g \in G} |\{x \in X : gx = x\}|$.
   [Answer: 6+2+4+4 = 16]

   (d) For each $x \in X$, list $\{g \in G : gx = x\}$.
   [Answer:

   - $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $\{e, r^2, rf, r^3 f\}$
   - $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $\{e, r^2, rf, r^3 f\}$
   - $x = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$: $\{e, r^3 f\}$
   - $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: $\{e, rf\}$
   - $x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$: $\{e, r^3 f\}$
   - $x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$: $\{e, rf\}$

   (e) Find $\sum_{x \in X} |\{g \in G : gx = x\}|$.
   [Answer: 4+4+2+2+2+2 = 16]

   (f) Find $|\{(g, x) \in G \times X : gx = x\}|$.
   [Answer: The set has 16 elements.]

3. In this set of exercises we walk through the first part proof with a specific example. Let D_4 act on the set of all two-by-two matrices with entries in $\mathbb{Z}_2$. Assume that f reflects a matrix about a vertical axis of symmetry and r rotates a matrix by ninety degrees.

   (a) Find $|X^g|$ for each $g \in G$.
   [Answer: $X^e = X$, $X^r = X^{r^3} = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$, $X^{r^2} = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$, $X^f = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$, $X^{r^2f} = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$, $X^{rf} = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$, $X^{r^3rf} = \left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$.]

   (b) For each $g \in G$, list $\{x \in X : gx = x\}$.
   [Answer:

   - $g = e$: X
   - $g = r$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$
   - $g = r^2$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix},\right\}$
   - $g = r^3$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$
   - $g = f$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$
   - $g = rf$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$
   - $g = r^2f$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$
   - $g = r^3f$: $\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$]

   (c) Find $\sum_{g\in G} |\{x \in X : gx = x\}|$.
   [Answer: 16+2+4+2+4+8+4+8 = 48]

   (d) For each $x \in X$, list $\{g \in G : gx = x\}$.
   [Answer:

   - $x = \begin{bmatrix}0&0\\0&0\end{bmatrix}$: G
   - $x = \begin{bmatrix}1&0\\0&0\end{bmatrix}$: $\{e, r^3f\}$
   - $x = \begin{bmatrix}0&1\\0&0\end{bmatrix}$: $\{e, rf\}$
   - $x = \begin{bmatrix}0&0\\0&1\end{bmatrix}$: $\{e, r^3f\}$

- $x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$: $\{e, rf\}$
- $x = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$: $\{e, f\}$
- $x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$: $\{e, r^2 f\}$
- $x = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$: $\{e, f\}$
- $x = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$: $\{e, r^2 f\}$
- $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $\{e, r^2, rf, r^3 f\}$
- $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$: $\{e, r^2, rf, r^3 f\}$
- $x = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$: $\{e, r^3 f\}$
- $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: $\{e, rf\}$
- $x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$: $\{e, r^3 f\}$
- $x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$: $\{e, rf\}$
- $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: G]

(e) Find $\sum_{x \in X} |\{g \in G : gx = x\}|$.
[Answer: 8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8 = 48]

(f) Find $|\{(g, x) \in G \times X : gx = x\}|$.
[Answer: The set has 48 elements.]

Chapter 4

Pólya Enumeration Questions

4.1 Necklaces

1. Find the cycle index for the dihedral group D_3 acting on the set of all necklaces with three beads.
   [Answer: $\frac{1}{6}(2a_3 + 3a_1a_2 + a_1^3)$.]

2. Use the cycle index to find the number of necklaces, up to rotations and reflections, on beads of two colors.
   [Answer: $\frac{1}{6}(2(2) + 3(2)(2) + (2)^3) = 4$.]

3. Use the cycle index to find the number of necklaces, up to rotations and reflections, on beads of three colors.
   [Answer: $\frac{1}{6}(2(3) + 3(3)(3) + (3)^3) = 10$.]

4. Use the cycle index to find the number of necklaces, up to rotations and reflections, on beads of n colors.
   [Answer: $\frac{1}{6}(2n + 3n^2 + n^3) = \frac{1}{6}n(n+2)(n+1)$. This is sequence A000292 in the OEIS.]

5. Find a generating function counting the number of three beaded necklaces, up to rotations and reflections, with red, white and black beads.
   [Answer: Setting $a_1 = (r + w + b), a_2 = (r^2 + w^2 + b^2)$, and $a_3 = (r^3 + w^3 + b^3)$ we expand to get $\frac{1}{6}(2(r^3 + w^3 + b^3) + 3(r + w + b)(r^2 + w^2 + b^2) + (r + w + b)^3) = b^3 + b^2r + b^2w + br^2 + brw + bw^2 + r^3 + r^2w + rw^2 + w^3$.]

6. How many necklaces are there, up to rotations and reflections, with exactly one red, one black, and one white bead?
   [Answer: The coefficient of brw is one, so there is just one such necklace.]

7. Find the cycle index for the cyclic group $\mathbb{Z}_3$ acting on the set of all necklaces with three beads.
   [Answer: $\frac{1}{3}(2a_3 + a_1^3)$.]

8. Use the cycle index to find the number of necklaces, up to rotations alone, on beads of two colors.
   [Answer: $\frac{1}{3}(2(2) + (2)^3) = 4$.]

9. Find a generating function counting the number of three beaded necklaces, up to rotations alone, with red, white and black beads.
[Answer: Setting $a_1 = (r + w + b), a_2 = (r^2 + w^2 + b^2)$ and $a_3 = (r^3 + w^3 + b^3)$, we expand to get $\frac{1}{3}(2(r^3 + w^3 + b^3) + (r + w + b)^3) = b^3 + b^2r + b^2w + br^2 + 2brw + bw^2 + r^3 + r^2w + rw^2 + w^3$.]

10. How many necklaces are there, up to rotations alone, with exactly one red, one black, and one white bead?
[Answer: The coefficient of brw is two, so there are two such necklace.]

11. Find the cycle index for the dihedral group D_4 acting on the set of all necklaces with four beads.
[Answer: $\frac{1}{8}(2a_4 + 3a_2^2 + 2a_1^2a_2 + a_1^4)$.]

12. Use the cycle index to find the number of four beaded necklaces, up to rotations and reflections, with beads of two colors.
[Answer: $\frac{1}{8}(2(2) + 3(2)^2 + 2(2)^2(2) + (2)^4) = 6$.]

13. Use the cycle index to find the number of four beaded necklaces, up to rotations and reflections, with beads of n colors.
[Answer: $\frac{1}{8}(2n + 3n^2 + 2n^3 + n^4) = \frac{1}{8}n(n + 1)(n^2 + n + 2)$. This is sequence A002817 in the OEIS.]

14. Find a generating function counting the number of four beaded necklaces, up to rotations and reflections, with blue and red beads.
[Answer: Setting $a_1 = (r + b), a_2 = (r^2 + b^2), a_3 = (r^3 + b^3)$ and $a_4 = (r^4 + b^4)$, we expand to get $\frac{1}{8}(2(r^4 + b^4) + 3(r^2 + b^2)^2 + 2(r + b)^2(r^2 + b^2) + (r + b)^4) = b^4 + b^3r + 2b^2r^2 + br^3 + r^4$.]

15. How many four beaded necklaces, up to rotations and reflections, have exactly two red and two blue beads?
[Answer: The coefficient of b^2r^2 is two, so there are two such necklaces.]

16. Find a generating function counting the number of four beaded necklaces, up to rotations and reflections, with blue, green and red beads.
[Answer: Setting $a_1 = (r + b + g), a_2 = (r^2 + b^2 + g^2), a_3 = (r^3 + b^3 + g^3)$ and $a_4 = (r^4 + b^4 + g^4)$, we expand to get $\frac{1}{8}(2(r^4 + b^4 + g^4) + 3(r^2 + b^2 + g^2)^2 + 2(r + b + g)^2(r^2 + b^2 + g^2) + (r + b + g)^4) = b^4 + b^3g + b^3r + 2b^2g^2 + 2b^2gr + 2b^2r^2 + bg^3 + 2bg^2r + 2bgr^2 + br^3 + g^4 + g^3r + 2g^2r^2 + gr^3 + r^4$.]

17. How many four beaded necklaces, up to rotations and reflections,have exactly two red, one green and one blue bead?
[Answer: The coefficient of r^2bg is two, so there are two such necklaces.]

18. Find a generating function counting the number of four beaded necklaces, up to rotations and reflections, with blue, yellow, green and red beads.
[Answer: Setting $a_1 = (r + b + g + y), a_2 = (r^2 + b^2 + g^2 + y^2), a_3 = (r^3 + b^3 + g^3 + y^3)$ and $a_4 = (r^4 + b^4 + g^4 + y^4)$, we expand to get $\frac{1}{8}(2(r^4 + b^4 + g^4 + y^4) + 3(r^2 + b^2 + g^2 + y^2)^2 + 2(r + b + g + y)^2(r^2 + b^2 + g^2 + y^2) + (r + b + g + y)^4) = b^4 + b^3g + b^3r + b^3y + 2b^2g^2 + 2b^2gr + 2b^2gy + 2b^2r^2 + 2b^2ry + 2b^2y^2 + bg^3 + 2bg^2r + 2bg^2y + 2bgr^2 + 3bgry + 2bgy^2 + br^3 + 2br^2y + 2bry^2 + by^3 + g^4 + g^3r + g^3y + 2g^2r^2 + 2g^2ry + 2g^2y^2 + gr^3 + 2gr^2y + 2gry^2 + gy^3 + r^4 + r^3y + 2r^2y^2 + ry^3 + y^4$.]

19. How many four beaded necklaces, up to rotations and reflections, have one red, one green, one yellow and one blue bead?
[Answer: The coefficient of $rbgy$ is three, so there are three such necklaces.]

20. Find the cycle index for the cyclic group $\mathbb{Z}_4$ acting on the set of all necklaces with four beads.
[Answer: $\frac{1}{4}(2a_4 + a_2^2 + a_1^4)$.]

21. Use the cycle index to find the number of four beaded necklaces, up to rotations alone, with beads of n colors.
[Answer: $\frac{1}{4}(2n + n^2 + n^4) = \frac{1}{4}n(n+1)(n^2 - n + 2)$. This is sequence A006528 in the OEIS.]

22. Find a generating function counting the number of four beaded necklaces, up to rotations alone, with blue and red beads.
[Answer: Setting $a_1 = (r+b), a_2 = (r^2+b^2)$, and $a_4 = (r^4+b^4)$, we expand to get $\frac{1}{4}(2(r^4+b^4)+(r^2+b^2)^2+(r+b)^4) = b^4+b^3r+2b^2r^2+br^3+r^4$.]

23. How many four beaded necklaces, up to rotations alone, have exactly two red and two blue beads?
[Answer: The coefficient of b^2r^2 is two, so there are two such necklaces.]

24. Find a generating function counting the number of four beaded necklaces, up to rotations alone, with blue, green and red beads.
[Answer: Setting $a_1 = (r+b+g), a_2 = (r^2+b^2+g^2)$, and $a_4 = (r^4+b^4+g^4)$, we expand to get $\frac{1}{4}(2(r^4+b^4+g^4)+(r^2+b^2+g^2)^2+(r+b+g)^4) = b^4+b^3g+b^3r+2b^2g^2+3b^2gr+2b^2r^2+bg^3+3bg^2r+3bgr^2+br^3+g^4+g^3r+2g^2r^2+gr^3+r^4$.]

25. How many four beaded necklaces, up to rotations alone, have exactly two red, one green and one blue bead?
[Answer: The coefficient of bgr^2 is three, so there are three such necklaces.]

26. How many four beaded necklaces, up to rotations alone, have one red, one green, one yellow and one blue bead?
[Answer: Setting $a_1 = (r+b+g+y), a_2 = (r^2+b^2+g^2+y^2)$, and $a_4 = (r^4+b^4+g^4+y^4)$, we expand to get $\frac{1}{4}(2(r^4+b^4+g^4+y^4)+(r^2+b^2+g^2+y^2)^2+(r+b+g+y)^4) = b^4+b^3g+b^3r+b^3y+2b^2g^2+3b^2gr+3b^2gy+2b^2r^2+3b^2ry+2b^2y^2+bg^3+3bg^2r+3bg^2y+3bgr^2+6bgry+3bgy^2+br^3+3br^2y+3bry^2+by^3+g^4+g^3r+g^3y+2g^2r^2+3g^2ry+2g^2y^2+gr^3+3gr^2y+3gry^2+gy^3+r^4+r^3y+2r^2y^2+ry^3+y^4$. The coefficient of $bgry$ is six, so there are six such necklaces.]

27. Find the cycle index for the dihedral group D_5 acting on the set of all necklaces with five beads.
[Answer: $\frac{1}{10}(4a_5 + 5a_1a_2^2 + a_1^5)$.]

28. Use the cycle index to find the number of five beaded necklaces, up to rotations and reflections, with beads of 2 colors.
[Answer: $\frac{1}{10}(4(2) + 5(2)(2)^2 + (2)^5) = 8$.]

29. Use the cycle index to find the number of five beaded necklaces, up to rotations and reflections, with two blue and three green beads.
[Answer: $\frac{1}{10}(4(b^5+g^5)+5(b+g)(b^2+g^2)^2+(b+g)^5) = b^5+b^4g+2b^3g^2+2b^2g^3+bg^4+g^5$. The coefficient of b^2g^3 is two so there are two such necklaces.]

30. Use the cycle index to find the number of five beaded necklaces, up to rotations and reflections, with beads of 3 colors.
[Answer: $\frac{1}{10}(4(3) + 5(3)(3)^2 + (3)^5) = 39$.]

31. Use the cycle index to find the number of five beaded necklaces, up to rotations and reflections, with beads of n colors.
[Answer: $\frac{1}{10}(4n + 5n^3 + n^5) = \frac{1}{10}n(n^2 + 4)(n^2 + 1)$. This is sequence A060446 in the OEIS.]

32. Find a generating function counting the number of five bead necklaces, up to rotations and reflections, with blue and green beads.
[Answer: We compute $\frac{1}{10}(4(b^5+g^5)+5(b+g)(b^2+g^2)^2+(b+g)^5)$ which is $b^5+b^4g+2b^3g^2+2b^2g^3+bg^4+g^5$.]

33. How many necklaces, up to rotations and reflections, have exactly two blue and three green beads?
[Answer: We take the coefficient of b^2g^3 to find our answer, which is two.]

34. Find a generating function counting the number of five bead necklaces, up to rotations and reflections, with blue, red and green beads.
[Answer: We expand $\frac{1}{10}(4(r^5 + b^5 + g^5) + 5(r + b + g)(r^2 + b^2 + g^2)^2 + (r + b + g)^5)$ which is $b^5 + b^4g + b^4r + 2b^3g^2 + 2b^3gr + 2b^3r^2 + 2b^2g^3 + 4b^2g^2r + 4b^2gr^2 + 2b^2r^3 + bg^4 + 2bg^3r + 4bg^2r^2 + 2bgr^3 + br^4 + g^5 + g^4r + 2g^3r^2 + 2g^2r^3 + gr^4 + r^5$.]

35. How many necklaces, up to rotations and reflections, have exactly one red, two blue and two green beads?
[Answer: We take the coefficient of b^2g^2r to find our answer, which is four.]

36. How many necklaces, up to rotations and reflections, have exactly one red bead, and four others which can be any combination of blue or green?
[Answer: We need to sum the coefficients of the terms that contain exactly one r. These terms are $b^4r, 2b^3gr, 4b^2g^2r, 2bg^3r, g^4r$. Their coefficients sum to ten, which is our answer.]

37. Find the cycle index for the cyclic group $\mathbb{Z}_5$ acting through rotation on the set of all necklaces with five beads.
[Answer: $\frac{1}{5}(4a_5 + a_1^5)$.]

38. Use the cycle index to find the number of five beaded necklaces, up to rotations alone, with two colored beads.
[Answer: $\frac{1}{5}(4(2) + (2)^5) = 8$.]

39. Use the cycle index to find the number of five beaded necklaces, up to rotations alone, with three colored beads.
[Answer: $\frac{1}{5}(4(3) + (3)^5) = 51$.]

40. Use the cycle index to find the number of five beaded necklaces, up to rotations alone, on five beaded necklaces in n colors.
[Answer: $\frac{1}{5}(4n + n^5) = \frac{1}{5}n(n^2 - 2n + 2)(n^2 + 2n + 2)$. This is sequence A054620 in the OEIS.]

41. Find a generating function counting the number of five bead necklaces, up to rotations alone, with red, blue and green beads.
[Answer: We expand $\frac{1}{5}(4(r^5 + b^5 + g^5) + (r + b + g)^5)$ to get $b^5 + b^4g + b^4r + 2b^3g^2 + 4b^3gr + 2b^3r^2 + 2b^2g^3 + 6b^2g^2r + 6b^2gr^2 + 2b^2r^3 + bg^4 + 4bg^3r + 6bg^2r^2 + 4bgr^3 + br^4 + g^5 + g^4r + 2g^3r^2 + 2g^2r^3 + gr^4 + r^5$.]

42. How many necklaces, up to rotations alone, have exactly one red, two blue and two green beads?
[Answer: We take the coefficient of b^2g^2r to find our answer, which is six.]

43. How many necklaces, up to rotations alone, have exactly one red bead, and four others which can be any combination of blue or green?
[Answer: We want the sum of the coefficients that contain exactly one r in our generating function. This sum is $1+4+6+4+1=16$ so our answer is sixteen.]

44. Find the cycle index for the dihedral group D_6 acting on the set of all necklaces with six beads.
[Answer: $\frac{1}{12}(2a_6+2a_3^2+4a_2^3+3a_1^2a_2^2+a_1^6)$.]

45. Use the cycle index to find the number of six beaded necklaces, up to rotations and reflections, with two colored beads.
[Answer: $\frac{1}{12}(2(2)+2(2)^2+4(2)^3+3(2)^2(2)^2+(2)^6)=13$.]

46. Use the cycle index to find the number of six beaded necklaces, up to rotations and reflections, with n colored beads.
[Answer: $\frac{1}{12}(2n+2n^2+4n^3+3n^2n^2+n^6)=\frac{1}{12}n(n+1)(n^4-n^3+4n^2+2)$. This is sequence A027670 in the OEIS.]

47. Find a generating function counting the number of six bead necklaces, up to rotations and reflections, with blue and purple beads.
[Answer: We expand $\frac{1}{12}(2(b^6+p^6)+2(b^3+p^3)^2+4(b^2+p^2)^3+3(b+p)^2(b^2+p^2)^2+(b+p)^6)$ to get $b^6+b^5p+3b^4p^2+3b^3p^3+3b^2p^4+bp^5+p^6$.]

48. Find the number of necklaces, up to rotations and reflections, with three blue and three purple beads.
[Answer: We take the coefficient of b^3p^3. This gives us three as our answer.]

49. Find a generating function counting the number of six bead necklaces, up to rotations and reflections, with blue, red and purple beads.
[Answer: We expand $\frac{1}{12}(2(b^6+p^6+r^6)+2(b^3+p^3+r^3)^2+4(b^2+p^2+r^2)^3+3(b+p+r)^2(b^2+p^2+r^2)^2+(b+p+r)^6)$ to get $b^6+b^5p+b^5r+3b^4p^2+3b^4pr+3b^4r^2+3b^3p^3+6b^3p^2r+6b^3pr^2+3b^3r^3+3b^2p^4+6b^2p^3r+11b^2p^2r^2+6b^2pr^3+3b^2r^4+bp^5+3bp^4r+6bp^3r^2+6bp^2r^3+3bpr^4+br^5+p^6+p^5r+3p^4r^2+3p^3r^3+3p^2r^4+pr^5+r^6$.]

50. Find the number of necklaces, up to rotations and reflections, with two red, two blue and two purple beads.
[Answer: We take the coefficient of $b^2p^2r^2$. This gives us eleven such necklaces.]

51. Find the number of necklaces, up to rotations and reflections, with one red, two blue and three purple beads.
[Answer: We take the coefficient of b^2p^3r which is six. Thus there are six such necklaces.]

52. Find the cycle index for the cyclic group $\mathbb{Z}_6$ acting through rotation on the set of all necklaces with six beads.
[Answer: $\frac{1}{6}(2a_6+2a_3^2+a_2^3+a_1^6)$.]

53. Use the cycle index to find the number of six beaded necklaces, up to rotations alone, with two colored beads.
[Answer: $\frac{1}{6}(2(2)+2(2)^2+(2)^3+(2)^6)=14$.]

54. Use the cycle index to find the number of six beaded necklaces, up to rotations alone, with n colored beads.
[Answer: $\frac{1}{6}(2n+2n^2+n^3+n^6)=\frac{1}{6}n(n+1)(n^4-n^3+n^2+2)$. This is sequence A006565 in the OEIS.]

55. Find a generating function counting the number of six bead necklaces, up to rotations alone, with red, blue and purple beads.
[Answer: We expand $\frac{1}{6}(2(r^6+b^6+p^6)+2(r^3+b^3+p^3)^2+(r^2+b^2+p^2)^3+(r+b+p)^6)$ to get $b^6+b^5p+b^5r+3b^4p^2+5b^4pr+3b^4r^2+4b^3p^3+10b^3p^2r+10b^3pr^2+4b^3r^3+3b^2p^4+10b^2p^3r+16b^2p^2r^2+10b^2pr^3+3b^2r^4+bp^5+5bp^4r+10bp^3r^2+10bp^2r^3+5bpr^4+br^5+p^6+p^5r+3p^4r^2+4p^3r^3+3p^2r^4+pr^5+r^6$.]

56. Find the number of necklaces, up to rotations alone, with two red, two blue and two purple beads.
[Answer: We take the coefficient of $b^2p^2r^2$. This gives us sixteen such necklaces.]

57. Find the cycle index for the dihedral group D_7 acting on the set of all necklaces with seven beads.
[Answer: $\frac{1}{14}(6a_7+7a_1a_2^3+a_1^7)$.]

58. Use the cycle index to find the number of seven beaded necklaces, up to rotations and reflections, with two colored beads.
[Answer: $\frac{1}{14}(6\times 2+7\times 2^4+2^7)=18$.]

59. Use the cycle index to find the number of seven beaded necklaces, up to rotations and reflections, with n colored beads.
[Answer: $\frac{1}{14}(6\times n+7\times n^4+n^7)=\frac{1}{14}n(n^3+6)(n+1)(n^2-n+1)$. This is sequence A060532 in the OEIS.]

60. Find the cycle index for the cyclic group $\mathbb{Z}_7$ acting through rotation on the set of all necklaces with seven beads.
[Answer: $\frac{1}{7}(6a_7+a_1^7)$.]

61. Use the cycle index to find the number of seven beaded necklaces, up to rotations alone, with two colored beads.
[Answer: $\frac{1}{7}(6(2)+(2)^7)=20$.]

62. Use the cycle index to find the number of seven beaded necklaces, up to rotations alone, with n colored beads.
[Answer: $\frac{1}{7}(6n+n^7)=\frac{1}{7}n(n^6+6)$. This is sequence A054621 in the OEIS.]

63. Find the cycle index for the dihedral group D_8 acting on the set of all necklaces with eight beads.
[Answer: $\frac{1}{16}(4a_8+2a_4^2+5a_2^4+4a_1^2a_2^3+a_1^8)$.]

64. Use the cycle index to find the number of eight beaded necklaces, up to rotations and reflections, with two colored beads.
[Answer: $\frac{1}{16}(4(2)+2(2)^2+5(2)^4+4(2)^2(2)^3+(2)^8)=30$.]

65. Use the cycle index to find the number of eight beaded necklaces, up to rotations and reflections, with n colored beads.
[Answer: $\frac{1}{16}(4n+2n^2+5n^4+4n^2n^3+n^8)=\frac{1}{16}n(n+1)(n^6-n^5+n^4+3n^3+2n^2-2n+4)$. This is sequence A060560 in the OEIS.]

66. Find the cycle index for the cyclic group $\mathbb{Z}_8$ acting through rotation on the set of all necklaces with eight beads.
[Answer: $\frac{1}{8}(4a_8+2a_4^2+a_2^4+a_1^8)$.]

67. Use the cycle index to find the number of eight beaded necklaces, up to rotations alone, with two colored beads.
[Answer: $\frac{1}{8}(4(2)+2(2)^2+(2)^4+(2)^8)=36$.]

68. Use the cycle index to find the number of eight beaded necklaces, up to rotations alone, with n colored beads.
[Answer: $\frac{1}{8}(4n+2n^2+n^4+n^8)=\frac{1}{8}n(n+1)(n^6-n^5+n^4-n^3+2n^2-2n+4)$. This is sequence A054622 in the OEIS.]

69. Find the cycle index for the dihedral group D_9 action on the set of all necklaces with nine beads.
[Answer: $\frac{1}{18}(6a_9+2a_3^3+9a_1a_2^4+a_1^9)$.]

70. Use the cycle index to find the number of nine beaded necklaces, up to rotations and reflections, with two colored beads.
[Answer: $\frac{1}{18}(6(2)+2(2)^3+9(2)(2)^4+(2)^9)=46$.]

71. Use the cycle index to find the number of nine beaded necklaces, up to rotations and reflections, with n colored beads.
[Answer: $\frac{1}{18}(6n+2n^3+9(n)n^4+n^9)=\frac{1}{18}n(n^8+9n^4+2n^2+6)$. This is sequence A060561 in the OEIS.]

72. Find the cycle index for the cyclic group $\mathbb{Z}_9$ acting through rotation on the set of all necklaces with nine beads.
[Answer: $\frac{1}{9}(6a_9+2a_3^3+a_1^9)$.]

73. Use the cycle index to find the number of nine beaded necklaces, up to rotations alone, with two colored beads.
[Answer: $\frac{1}{9}(6(2)+2(2)^3+(2)^9)=60$.]

74. Use the cycle index to find the number of nine beaded necklaces, up to rotations alone, with n colored beads.
[Answer: $\frac{1}{9}(6n+2n^3+n^9)=\frac{1}{9}n(n^8+2n^2+6)$. This is sequence A054623 in the OEIS.]

4.2 Graphs

1. Find the cycle index for the edge permutation group of the complete graph K_2. This means considering the group action of edge permutations that arise from all permutations of the two vertices.
[Answer: $\frac{1}{2}(2a_1)=a_1$.]

2. Use this to find the number of graphs up to isomorphism on two vertices.
[Answer: There is a bijection between complete graphs in two colors under this action and all graphs up to isomorphism, since we can consider edges on one color to be included and edges of another color to be not. Thus we put two into each a_i in our cycle index to get an answer of two.]

3. Find the cycle index for the edge permutation group of the complete graph K_3. This means considering the group action of edge permutations that arise from all permutations of the three vertices.
[Answer: $\frac{1}{6}(3a_1a_2+2a_3+a_1^3)$.]

4. Use this to find the number of graphs up to isomorphism on three vertices.
[Answer: $\frac{1}{6}(12 + 4 + 2^3) = 4$ so our answer is four.]

5. Find a generating function counting the number of graphs up to isomorphism with n edges on three vertices.
[Answer: Consider coloring an edge white to mean removing it. We can then just take the generating function that counts every possible combination of two colors by plugging $b^n + w^n$ for a_n. We get $\frac{1}{6}(3(b+w)(b^2+w^2) + 2(b^3+w^3) + (b+w)^3) = b^3 + b^2w + bw^2 + w^3$. To make things look even nicer, and as the number of not included edges is determined by the number of included ones, we can just plug in one for w to get $b^3 + b^2 + b + 1$. Here the coefficient of b^n is the number of graphs with n edges. Note that there is only one possibility for each number of edges. This is because there isn't much room with only three vertices. Any two edges must be adjacent, and any three edges must form a coput of K_3 itself.]

6. Find the cycle index for the edge permutation group of the complete graph K_4.
[Answer: $\frac{1}{24}(6a_2^2a_1^2 + 8a_3^2 + 3a_2^2a_1^2 + 6a_4a_2 + a_1^6)$.]

7. Use this to find the number of graphs up to isomorphism on four vertices.
[Answer: $\frac{1}{24}(6(2^4) + 8(2^2) + 3(2^4) + 6(2^2) + 2^6) = 11$ which means there are eleven graphs up to isomorphism.]

8. Find a generating function counting the number of graphs up to isomorphism with n edges on four vertices.
[Answer: We get $\frac{1}{24}(6(b^2+w^2)^2(b+w)^2+8(b^3+w^3)^2+3(b^2+w^2)^2(b+w)^2+6(b^4+w^4)(b^2+w^2)+(b+w)^6) = b^6 + b^5w + 2b^4w^2 + 3b^3w^3 + 2b^2w^4 + bw^5 + w^6$ Plugging one for w gives $b^6 + b^5 + 2b^4 + 3b^3 + 2b^2 + b + 1$. Here the coefficient of b^n is the number of graphs with n edges.]

9. How many graphs are there up to isomorphism with three edges on four vertices?
[Answer: We simply read off the coefficient of b^3 which is three.]

10. Find the cycle index for the edge permutation group of the complete graph K_5.
[Answer: $\frac{1}{120}(10a_2^3a_1^4 + 20a_3^3a_1 + 15a_2^4a_1^2 + 30a_4^2a_2 + 20a_6a_3a_1 + 24a_5^2 + a_1^{10})$.]

11. Use this to find the number of graphs up to isomorphism on five vertices.
[Answer: $\frac{1}{120}(10 \times 2^7 + 20 \times 2^4 + 15 \times 2^6 + 30 \times 2^3 + 20 \times 2^3 + 24 \times 2^2 + 2^{10}) = 34$ which means there are thirty-four graphs up to isomorphism.]

12. Find a generating function counting the number of graphs up to isomorphism with n edges on five vertices.
[Answer: Again, we consider the generating function which assigns one of two colors to the edges, and consider coloring an edge in one of the colors the same as removing it. We take the generating function that counts this by plugging $b^n + w^n$ for a_n. We get $\frac{1}{120}(10(b^2+w^2)^3(b+w)^4 + 20(b^3+w^3)^3(b+w) + 15(b^2+w^2)^4(b+w)^2 + 30(b^4+w^4)^2(b^2+w^2) + 20(b^6+w^6)(b^3+w^3)(b+w) + 24(b^5+w^5)^2 + (b+w)^{10}) = b^10 + b^9w + 2b^8w^2 + 4b^7w^3 + 6b^6w^4 + 6b^5w^5 + 6b^4w^6 + 4b^3w^7 + 2b^2w^8 + bw^9 + w^10$. We can again plug one in for w to get $b^10 + b^9 + 2b^8 + 4b^7 + 6b^6 + 6b^5 + 6b^4 + 4b^3 + 2b^2 + b + 1$. Here the coefficient of b^n is the number of graphs with n edges.]

13. How many graphs up to isomorphism are there with three edges on five vertices?
[Answer: We take the coefficient of b^3 which is four to get our answer. Notice that because there are

only five vertices, it is impossible to get the graph with three edges all non-adjacent to each other. For n greater than five, that then becomes possible so the number will max out at five possibilities.]

14. Find the cycle index for the edge permutation group of the complete directed graph on two vertices. Thus here we have an S_2 action on the set of two directed edges, where each permutation of edges arises from a permutation of the vertices.
[Answer: $\frac{1}{2}(a_2 + a_1^2)$.]

15. Use this to find the number of directed graphs up to isomorphism on two vertices.
[Answer: $\frac{1}{2}(2 + 2^2) = 3$ which means there are three different graphs.]

16. Recall that an irreflexive relation is one where no element relates to itself. Find the number of irreflexive relations up to isomorphism on a set of two elements.
[Answer: These have no loops and thus are in correspondence with directed graphs. Therefore there are three irreflexive relations.]

17. Recall that a reflexive relation is one where each element relates to itself. Find the number of reflexive relations up to isomorphism on a set of two elements.
[Answer: These are in one-to-one correspondence with the set of irreflexive relations because the map that adds loops to every vertex is a bijection between them. Therefore there are three reflexive relations.]

18. Find the cycle index for the edge permutation group of the complete directed graph on three vertices
[Answer: $\frac{1}{6}(3a_2^3 + 2a_3^2 + a_1^6)$.]

19. Use this to find the number of directed graphs up to isomorphism on three vertices.
[Answer: $\frac{1}{6}(3 \times 2^3 + 2 \times 2^2 + 2^6) = 16$ which means there are sixteen different directed graphs. This is also the number of both reflexive and irreflexive relations up to isomorphism on three elements.]

20. Find the cycle index for the edge permutation group of the complete directed graph on four vertices.
[Answer: $\frac{1}{24}(6a_2^5a_1^2 + 8a_3^4a_1^4 + 3a_2^6 + 6a_4^3 + a_1^{12})$.]

21. Use this to find the number of directed graphs up to isomorphism on four vertices.
[Answer: $\frac{1}{24}(6(2^7) + 8(2^4) + 3(2^6) + 6(2^3) + 2^{12}) = 218$ which means there are 218 different directed graphs. This is also the number of both reflexive and irreflexive relations up to isomorphism on four elements.]

22. Find the cycle index for the edge permutation group of the complete directed graph on five vertices.
[Answer: $\frac{1}{120}(10a_2^7a_1^6 + 20a_3^6a_1^2 + 15a_2^{10} + 30a_4^5 + 20a_6^2a_3^2a_2 + 24a_5^4 + a_1^{20})$.]

23. Use this to find the number of directed graphs up to isomorphism on five vertices.
[Answer:$\frac{1}{120}(10 \times 2^{13} + 20 \times 2^8 + 15 \times 2^{10} + 30 \times 2^5 + 20 \times 2^5 + 24 \times 2^4 + 2^{20}) = 9608$. Thus there are 9608 different directed graphs. This is also the number of both reflexive and irreflexive relations up to isomorphism on five elements.]

24. Find the cycle index for the edge permutation group of the complete directed graph with loops on two vertices.
[Answer: $\frac{1}{2}(a_2^2 + a_1^4)$.]

25. Use this to find the number of relations up to isomorphism on two vertices.
[Answer: $\frac{1}{2}(2^2 + 2^4) = 10$ which means there are ten different relations.]

26. Find the cycle index for the edge permutation group of the complete directed graph with loops on three vertices.
[Answer: $\frac{1}{6}(3a_2^4a_1 + 2a_3^3 + a_1^9)$.]

27. Use this to find the number of relations up to isomorphism on three vertices.
[Answer: $\frac{1}{6}(3 \times 2^5 + 2 \times 2^3 + 2^9) = 104$ which means there are 104 different relations.]

28. Find the cycle index for the edge permutation group of the complete directed graph with loops on four vertices.
[Answer: $\frac{1}{24}(6a_2^6a_1^4 + 8a_3^5a_1 + 3a_2^8 + 6a_4^4 + a_1^{16})$.]

29. Use this to find the number of relations up to isomorphism on four vertices.
[Answer:$\frac{1}{24}(6 \times 2^{10} + 8 \times 2^6 + 3 \times 2^8 + 6 \times 2^4 + 2^{16}) = 3044$.]

30. Find the cycle index for the edge permutation group of the complete directed graph with loops on five vertices.
[Answer: $\frac{1}{120}(10a_2^8a_1^9 + 20a_3^7a_1^4 + 15a_2^{12}a_1 + 30a_4^6a_1 + 20a_6^2a_3^3a_2^2 + 24a_5^5 + a_1^{25})$.]

31. Use this to find the number of relations up to isomorphism on five vertices.
[Answer:$\frac{1}{120}(10 \times 2^{17} + 20 \times 2^{11} + 15 \times 2^{13} + 30 \times 2^7 + 20 \times 2^7 + 24 \times 2^5 + 2^{25}) = 291968$. Thus there are 291968 possible relations on a set of five elements.]

32. Find the cycle index for the edge permutation group of the complete graph with loops on two vertices.
[Answer: $\frac{1}{2}(a_1a_2 + a_1^3)$.]

33. Use this to find the number of symmetric relations up to isomorphism on a set of two elements.
[Answer: The symmetric relations are in one-to-one correspondence with the set of (not-directed) graphs with possible loops. We know there are $\frac{1}{2}(4 + 8)$ of these so our answer is six.]

34. Find the cycle index for the edge permutation group of the complete graph with loops on three vertices.
[Answer: $\frac{1}{6}(3a_1^2a_2^2 + 2a_3^2 + a_1^6)$.]

35. Use this to find the number of symmetric relations up to isomorphism on a set of three elements.
[Answer: $\frac{1}{6}(3 \times 16 + 2 \times 4 + 2^6) = 20$ so our answer is twenty.]

36. Find the cycle index for the edge permutation group of the complete graph with loops on four vertices.
[Answer: $\frac{1}{24}(6a_2^3a_1^4 + 8a_3^3a_1 + 3a_2^4a_1^2 + 6a_4^2a_2 + a_1^{10})$.]

37. Use this to find the number of symmetric relations up to isomorphism on a set of four elements.
[Answer: $\frac{1}{24}(6 \times 2^7 + 8 \times 2^4 + 3 \times 2^6 + 6 \times 2^3 + 2^{10}) = 90$ which means there are ninety symmetric relations.]

38. Find the cycle index for the edge permutation group of the complete graph with loops on five vertices.
[Answer: $\frac{1}{120}(10a_2^4a_1^7 + 20a_3^4a_1^3 + 15a_2^6a_1^3 + 30a_4^3a_2a_1 + 20a_6a_3^2a_2a_1 + 24a_5^3 + a_1^{15})$.]

39. Use this to find the number of symmetric relations up to isomorphism on a set of five elements.
[Answer: $\frac{1}{120}(10 \times 2^{11} + 20 \times 2^7 + 15 \times 2^9 + 30 \times 2^5 + 20 \times 2^5 + 24 \times 2^3 + 2^{15}) = 544$ which means there are 544 symmetric relations on a set of five elements.]

4.3 Platonic Solids

1. Find the cycle index for the corner permutation group of the tetrahedron under the action of orientation preserving rotations.
   [Answer: We know the group of rotations fixing a tetrahedron gives us an A_4 action on the set of corners. This gives us $\frac{1}{12}(8a_1a_3 + 3a_2^2 + a_1^4)$.]

2. How many distinct ways can we color the corners of a tetrahedron with two colors?
   [Answer: We set each a_i equal to two to get $\frac{1}{12}(8(2)(2) + 3(2^2) + 2^4) = 5$. Thus there are five total ways.]

3. How many distinct ways can we color the corners of a tetrahedron with n colors?
   [Answer: We set each a_i equal to n to get $\frac{1}{12}(11n^2 + n^4) = \frac{1}{12}n^2(n^2 + 11)$. This is sequence A006008 in the OEIS.]

4. Find the cycle index for the edge permutation group of the tetrahedron.
   [Answer: We again have an A_4 action on the set of edges. We get $\frac{1}{12}(8a_3^2 + 3a_2^2a_1^2 + a_1^6)$.]

5. How many distinct ways can we color the edges of a tetrahedron with two colors?
   [Answer: We set each a_i equal to two to get $\frac{1}{12}(8 \times 2^2 + 3 \times 2^4 + 2^6) = 12$. Thus there are twelve total ways.]

6. How many distinct ways can we color the edges of a tetrahedron with n colors?
   [Answer: We set each a_i equal to n to get $\frac{1}{12}(8 \times n^2 + 3 \times n^4 + n^6) = \frac{1}{12}n^2(n^4 + 3n^2 + 8)$. This is sequence A046023 in the OEIS.]

7. Find a generating function for the number of edge colored tetrahedra in two colors.
   [Answer: Naming our the colors red and blue, we substitute $(r^n + b^n)$ into each a_n to get $\frac{1}{12}(8(r^3 + b^3)^2 + 3(r^2 + b^2)^2(r + b)^2 + (r + b)^6) = b^6 + b^5r + 2b^4r^2 + 4b^3r^3 + 2b^2r^4 + br^5 + r^6$.]

8. How many ways can we color the edges of a tetrahedron in two colors so that we have the same number of edges of each color?
   [Answer: We look at the coefficient of b^3r^3 to get our answer of four.]

9. Find a generating function for the number of edge colored tetrahedra in three colors.
   [Answer: Name the colors red, blue, and green. We substitute $(r^n + b^n + g^n)$ into our a_n to get $\frac{1}{12}(8(r^3 + b^3 + g^3)^2 + 3(r^2 + b^2 + g^2)^2(r + b + g)^2 + (r + b + g)^6) = b^6 + b^5g + b^5r + 2b^4g^2 + 3b^4gr + 2b^4r^2 + 4b^3g^3 + 6b^3g^2r + 6b^3gr^2 + 4b^3r^3 + 2b^2g^4 + 6b^2g^3r + 9b^2g^2r^2 + 6b^2gr^3 + 2b^2r^4 + bg^5 + 3bg^4r + 6bg^3r^2 + 6bg^2r^3 + 3bgr^4 + br^5 + g^6 + g^5r + 2g^4r^2 + 4g^3r^3 + 2g^2r^4 + gr^5 + r^6$.]

10. How many edge colored tetrahedra have exactly two red, two, blue and two green edges?
    [Answer: We look at the coefficient of $b^2g^2r^2$ to get an answer of nine.]

11. Find the cycle index for the face permutation group of the tetrahedron.
    [Answer: This is the same as the corner permutation group as the tetrahedron is self dual.]

12. Find the cycle index for the corner permutation group of the cube.
    [Answer: We have S_4 permuting the eight corners of the cube. We get $\frac{1}{24}(8a_3^2a_1^2 + 6a_4^2 + 3a_2^4 + 6a_2^4 + a_1^8)$.]

13. How many distinct ways can we color the corners of a cube in two colors?
    [Answer: $\frac{1}{24}(8 \times 2^4 + 6 \times 2^2 + 3 \times 2^4 + 6 \times 2^4 + 2^8) = 23$. Thus there are twenty-three possible colorings.]

14. How many distinct ways can we color the corners of a cube in n colors?
[Answer: $\frac{1}{24}(8\times n^4+6\times n^2+3\times n^4+6\times n^4+n^8)=\frac{1}{24}n^2(n^6+17n^2+6)$. This is sequence A000543 in the OEIS.]

15. Find a generating function for the number of cubes with corners of two colors.
[Answer: $\frac{1}{24}(8(b^3+w^3)^2(b+w)^2+6(b^4+w^4)^2+3(b^2+w^2)^4+6(b^2+w^2)^4+(b+w)^8)=b^8+b^7w+3b^6w^2+3b^5w^3+7b^4w^4+3b^3w^5+3b^2w^6+bw^7+w^8$.]

16. How many ways can we color the corner of cubes in two colors, so each color is used the same number of times?
[Answer: We take the coefficient of b^4w^4 to get our answer of seven.]

17. How many ways can we color the corner of cubes in white and black, so more corners are black than white?
[Answer: There are twenty-three colorings in total. Seven have the same number of white and black corners. In the remaining sixteen, just as many will have more black than white as those with more white than black. Thus we get half of sixteen or eight total colorings. We could have also arrived at this answer by adding the coefficients of b^8, b^7w, b^6w^2, and b^5w^3 to arrive at $1+1+3+3=8$, which gives us the same answer.]

18. Find a generating function for the number of cubes with corners of three colors.
[Answer: $\frac{1}{24}(8(r^3+b^3+g^3)^2(r+b+g)^2+6(r^4+b^4+g^4)^2+3(r^2+b^2+g^2)^4+6(r^2+b^2+g^2)^4+(r+b+g)^8)=b^8+b^7g+b^7r+3b^6g^2+3b^6gr+3b^6r^2+3b^5g^3+7b^5g^2r+7b^5gr^2+3b^5r^3+7b^4g^4+13b^4g^3r+22b^4g^2r^2+13b^4gr^3+7b^4r^4+3b^3g^5+13b^3g^4r+24b^3g^3r^2+24b^3g^2r^3+13b^3gr^4+3b^3r^5+3b^2g^6+7b^2g^5r+22b^2g^4r^2+24b^2g^3r^3+22b^2g^2r^4+7b^2gr^5+3b^2r^6+bg^7+3bg^6r+7bg^5r^2+13bg^4r^3+13bg^3r^4+7bg^2r^5+3bgr^6+br^7+g^8+g^7r+3g^6r^2+3g^5r^3+7g^4r^4+3g^3r^5+3g^2r^6+gr^7+r^8$.]

19. How many ways can we color the corners of the cube in red, blue and green so that the number of red corners equals the number of blue corners?
[Answer: We add up the coefficients of $b^4r^4, g^2b^3r^3, g^4b^2r^2, g^6br$, and g^8 to get $7+24+22+3+1=57$.]

20. Find the cycle index for the edge permutation group of the cube.
[Answer: We have S_4 permuting the twelve edges of the cube. We arrive at $\frac{1}{24}(8a_3^4+6a_4^3+3a_2^6+6a_2^5a_1^2+a_1^{12})$.]

21. How many ways can we color the edges of a cube with two colors?
[Answer: We get $\frac{1}{24}(8\times 2^4+6\times 2^3+3\times 2^6+6\times 2^7+2^{12})=218$. Therefore there are 218 colorings using two colors.]

22. How many ways can we color the edges of a cube with n colors?
[Answer: We get $\frac{1}{24}(8\times n^4+6\times n^3+3\times n^6+6\times n^7+n^{12})=\frac{1}{24}n^3(n^9+6n^4+3n^3+8n+6)$. This is sequence A060530 in the OEIS.]

23. Find a generating function for the number of cubes with edges of two colors.
[Answer: We get $\frac{1}{24}(8(b^3+w^3)^4+6(b^4+w^4)^3+3(b^2+w^2)^6+6(b^2+w^2)^5(b+w)^2+(b+w)^{12})=b^{12}+b^{11}w+5b^{10}w^2+13b^9w^3+27b^8w^4+38b^7w^5+48b^6w^6+38b^5w^7+27b^4w^8+13b^3w^9+5b^2w^{10}+bw^{11}+w^{12}$.]

24. How many edge colorings of the cube use two colors, and have both colors used the same number of times?
[Answer: We take the coefficient of b^6w^6 to get our answer of forty-eight.]

25. Find the cycle index for the face permutation group of the cube.
[Answer: We have S_4 permuting the six faces of the cube. We arrive at $\frac{1}{24}(8a_3^2+6a_4a_1^2+3a_2^2a_1^2+6a_2^3+a_1^6)$.]

26. How many ways can we color the faces of a cube with two colors?
[Answer: $\frac{1}{24}(8\times 2^2+6\times 2^3+3\times 2^4+6\times 2^3+2^6)=10$. Thus there are ten distinct ways to color the faces of a cube in two colors.]

27. How many ways can we color the faces of a cube with n colors?
[Answer: $\frac{1}{24}(8\times n^2+6\times n^3+3\times n^4+6\times n^3+n^6)=\frac{1}{24}n^2(n+1)(n^3-n^2+4n+8)$. This is sequence A047780 in the OEIS.]

28. Find the cycle index for the corner, edge and face permutation groups of the octahedron.
[Answer: Respectively, these are the same as the cycle index of the face, edge and corner permutation groups of the cube. These are all computed above.]

29. Find the cycle index for the corner permutation group of the dodecahedron.
[Answer: We have A_5 permuting the twenty corners of the dodecahedron. We get $\frac{1}{60}(20a_3^6a_1^2+24a_5^4+15a_2^{10}+a_1^{20})$.]

30. How many ways can we color the corners of a dodecahedron with two colors.
[Answer: $\frac{1}{60}(20\times 2^8+24\times 2^4+15\times 2^{10}+2^{20})=17824$.]

31. How many ways can we color the corners of a dodecahedron with n colors.
[Answer: $\frac{1}{60}(20\times n^8+24\times n^4+15\times n^{10}+n^{20})=\frac{1}{60}n^4(n^16+15n^6+20n^4+24)$. This is sequence A054472 in the OEIS.]

32. Find the cycle index for the edge permutation group of the dodecahedron.
[Answer: We have A_5 permuting the thirty edges of the dodecahedron. We arrive at $\frac{1}{60}(20a_3^{10}+24a_5^6+15a_2^{14}a_1^2+a_1^{30})$.]

33. How many ways can we color the edges of a dodecahedron with two colors.
[Answer: $\frac{1}{60}(20\times 2^{10}+24\times 2^6+15\times 2^{16}+2^{30})=17912448$. Thus there are 17912448 possible colorings.]

34. How many ways can we color the edges of a dodecahedron with n colors.
[Answer: $\frac{1}{60}(20\times n^{10}+24\times n^6+15\times n^{16}+n^{30})=\frac{1}{60}n^6(n^{24}+15n^{10}+20n^4+24)$. This is sequence A282670 in the OEIS.]

35. Find the cycle index for the face permutation group of the dodecahedron.
[Answer: We have A_5 permuting the twelve faces of the dodecahedron. We arrive at $\frac{1}{60}(20a_3^4+24a_5^2a_1^2+15a_2^6+a_1^{12})$.]

36. How many ways can we color the faces of a dodecahedron with two colors.
[Answer: $\frac{1}{60}(20\times 2^4+24\times 2^4+15\times 2^6+2^{12})=96$.]

37. How many ways can we color the faces of a dodecahedron with n colors.
[Answer: $\frac{1}{60}(20\times n^4+24\times n^4+15\times n^6+n^{12})=\frac{1}{60}n^4(n^8+15n^2+44)$. This is sequence A000545 in the OEIS.]

38. Find the cycle index for the corner, edge and face permutation groups of the icosahedron.
[Answer: Respectively, these are the same as the cycle index of the face, edge and corner permutation groups of the dodecahedron, which we computed above.]

4.4 Matrices

1. Consider a group action of $S_2 = \{e, t\}$ on the set of entries of a two by two matrix. Here e leaves matrices fixed and t maps a matrix to its transpose. Find the cycle index for this group of permutations on the set of four entries.
   [Answer: $\frac{1}{2}(a_2a_1^2 + a_1^4)$.]
2. Use this to find the number of matrices over the set $M_{2\times 2}(2)$ of two-by-two matrices over $\mathbb{Z}_2$, up to taking the transpose.
   [Answer: $\frac{1}{2}(2^3 + 2^4) = 12$. Thus there are twelve possible matrices up to this equivalence.]
3. Use this to find the number of matrices over the set $M_{2\times 2}(n)$ of two-by-two matrices over $\mathbb{Z}_n$, up to taking the transpose.
   [Answer: $\frac{1}{2}(n^3 + n^4) = \frac{1}{2}n^3(n + 1)$. This is sequence A019582 in the OEIS]
4. Consider a group action of $S_2 = \{e, f\}$ on the set of entries of a two by two matrix. Here e leaves matrices fixed and f switches the two rows. Find the cycle index for this group of permutations on the set of four entries.
   [Answer: $\frac{1}{2}(a_2^2 + a_1^4)$.]
5. Use this to find the number of matrices over the set $M_{2\times 2}(2)$ of two-by-two matrices over $\mathbb{Z}_2$, up to the action of switching the rows.
   [Answer: $\frac{1}{2}(2^2 + 2^4) = 10$. Thus there are ten distinct matrices under this equivalence.]
6. Use this to find the number of matrices over the set $M_{2\times 2}(n)$ of two-by-two matrices over $\mathbb{Z}_n$, up to the action of switching the rows.
   [Answer: $\frac{1}{2}(n^2 + n^4) = \frac{1}{2}n^2(n^2 + 1)$. This is sequence A037270 in the OEIS.]
7. Consider a group action of $\mathbb{Z}_4 = \{e, r, r^2, r^3\}$ on the set of entries of a two by two matrix. Here e leaves matrices fixed and r rotates the matrix ninety degrees, r^2 rotates twice, and r^3 does this three times. Find the cycle index for this group of permutations on the set of four matrix entries.
   [Answer: $\frac{1}{4}(2a_4 + a_2^2 + a_1^4)$.]
8. Use this to find the number of matrices over the set $M_{2\times 2}(2)$ of two-by-two matrices over $\mathbb{Z}_2$, up to the action of rotation.
   [Answer: $\frac{1}{4}(2 \times 2 + 2^2 + 2^4) = 6$. Thus there are six distinct matrices up to rotation.]
9. Use this to find the number of matrices over the set $M_{2\times 2}(n)$ of two-by-two matrices over $\mathbb{Z}_n$, up to the action of rotation.
   [Answer: $\frac{1}{4}(2n + n^2 + n^4) = \frac{1}{4}n(n + 1)(n^2 - n + 2)$. This is sequence A006528 in the OEIS.]
10. Consider a group action of $V = \{e, h, v, hv\}$ on the set of entries of a two by two matrix. Here e leaves matrices fixed and h reflects them across a horizontal line of symmetry, v reflects them across a vertical line of symmetry, and hv does both. Find the cycle index for this group of permutations on the set of four matrix entries.
    [Answer: $\frac{1}{4}(3a_2^2 + a_1^4)$.]
11. Use this to find the number of matrices over the set $M_{2\times 2}(2)$ of two-by-two matrices over $\mathbb{Z}_2$, up to the action of vertical and horizontal reflections.
    [Answer: $\frac{1}{4}(3 \times 2^2 + 2^4) = 7$. Thus there are seven distinct matrices up to our reflections.]

12. Use this to find the number of matrices over the set $M_{2\times 2}(n)$ of two-by-two matrices over $\mathbb{Z}_n$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(3 \times n^2 + n^4) = \frac{1}{4}n^2(n^2+3)$. This is sequence A039623 in the OEIS.]

13. Consider a group action of D_4 on the set of entries of a two by two matrix, acting upon them by rotations and reflections. Find the cycle index for this group of permutations on the set of four matrix entries.
[Answer: $\frac{1}{8}(2a_4 + 3a_2^2 + 2a_2a_1^2 + a_1^4)$.]

14. Use this to find the number of matrices over the set $M_{2\times 2}(2)$ of two-by-two matrices over $\mathbb{Z}_2$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2 \times 2 + 3 \times 2^2 + 2 \times 2 \times 2^2 + 2^4) = 6$. Thus there are six distinct matrices up to rotations and reflections.]

15. Use this to find the number of two-by-two matrices over the set $M_{2\times 2}(n)$ of two-by-two matrices over $\mathbb{Z}_n$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2 \times n + 3 \times n^2 + 2 \times n \times n^2 + n^4) = \frac{1}{8}n(n+1)(n^2+n+2)$. This is sequence A002817 in the OEIS.]

16. Consider a group action of $S_2 = \{e, t\}$ on the set of entries of a three-by-three matrix. Here e leaves matrices fixed and t maps a matrix to its transpose. Find the cycle index for this group of permutations on the set of nine entries.
[Answer: $\frac{1}{2}(a_2^3a_1^3 + a_1^9)$.]

17. Use this to find the number of matrices over the set $M_{3\times 3}(2)$ of three-by-three matrices over $\mathbb{Z}_2$, up to taking the transpose.
[Answer: $\frac{1}{2}(2^3 \times 2^3 + 2^9) = 288$. Thus there are 288 possible matrices up to this equivalence.]

18. Use this to find the number of matrices over the set $M_{3\times 3}(n)$ of three-by-three matrices over $\mathbb{Z}_n$, up to taking the transpose.
[Answer: $\frac{1}{2}(n^3 \times n^3 + n^9) = \frac{1}{2}n^6(n+1)(n^2-n+1)$. This is sequence A168555 in the OEIS.]

19. Consider a group action of S_3 on the set of entries of a three-by-three matrix acting by permuting the rows. Find the cycle index for this group of permutations on the set of nine entries.
[Answer: $\frac{1}{6}(3a_2^3a_1^3 + 2a_3^3 + a_1^9)$.]

20. Use this to find the number of matrices over the set $M_{3\times 3}(2)$ of three-by-three matrices over $\mathbb{Z}_2$, up to the action of permuting the rows.
[Answer: $\frac{1}{6}(3 \times 2^6 + 2 \times 2^3 + 2^9) = 120$. Thus there are ten distinct matrices under this equivalence.]

21. Use this to find the number of matrices over the set $M_{3\times 3}(n)$ of three-by-three matrices over $\mathbb{Z}_n$, up to the action of permuting the rows.
[Answer: $\frac{1}{6}(3 \times n^6 + 2 \times n^3 + n^9) = \frac{1}{6}n^3(n^3+2)(n+1)(n^2-n+1)$. This is sequence A282612 in the OEIS.]

22. Consider a group action of $\mathbb{Z}_4 = \{e, r, r^2, r^3\}$ on the set of entries of a three-by-three matrix. Here e leaves matrices fixed and r rotates the matrix ninety degrees, r^2 rotates twice, and r^3 does this three times. Find the cycle index for this group of permutations on the set of nine entries.
[Answer: $\frac{1}{4}(2a_4^2a_1 + a_2^4a_1 + a_1^9)$.]

23. Use this to find the number of matrices over the set $M_{3\times 3}(2)$ of three-by-three matrices over $\mathbb{Z}_2$, up to rotations.
[Answer: $\frac{1}{4}(2\times 2^3+2^5+2^9)=140$. Thus there are 140 distinct matrices up to rotation.]

24. Use this to find the number of matrices over the set $M_{3\times 3}(n)$ of three-by-three matrices over $\mathbb{Z}_n$, up to rotations.
[Answer: $\frac{1}{4}(2\times n^3+n^5+n^9)=\frac{1}{4}n^3(n^2+1)(n^4-n^2+2)$. This is sequence A282613 in the OEIS.]

25. Consider a group action of $V=\{e,h,v,hv\}$ on the set of entries of a three-by-three matrix. Here e leaves matrices fixed and h reflects them across a horizontal line of symmetry, v reflects them across a vertical line of symmetry, and hv does both. Find the cycle index for this group of permutations on the set of nine entries.
[Answer: $\frac{1}{4}(2a_2^3a_1^3+a_2^4a_1+a_1^9)$.]

26. Use this to find the number of matrices over the set $M_{3\times 3}(2)$ of three-by-three matrices over $\mathbb{Z}_2$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(2\times 2^6+2^5+2^9)=184$. Thus there are 184 distinct matrices up to our reflections.]

27. Use this to find the number of matrices over the set $M_{3\times 3}(n)$ of three-by-three matrices over $\mathbb{Z}_n$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(2\times n^6+n^5+n^9)=\frac{1}{4}n^5(n+1)(n^3-n^2+n+1)$. This is sequence A282614 in the OEIS.]

28. Consider a group action of D_4 on the set of entries of a three-by-three matrix, acting upon them by rotations and reflections. Find the cycle index for this group of permutations on the set of nine entries.
[Answer: $\frac{1}{8}(2a_4^2a_1+a_2^4a_1+4a_2^3a_1^3+a_1^9)$.]

29. Use this to find the number of matrices over the set $M_{3\times 3}(2)$ of three-by-three matrices over $\mathbb{Z}_2$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2\times 2^3+2^5+4\times 2^6+2^9)=102$. Thus there are 102 distinct matrices up to rotations and reflections.]

30. Use this to find the number of matrices over the set $M_{3\times 3}(n)$ of three-by-three matrices over $\mathbb{Z}_n$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2\times n^3+n^5+4\times n^6+n^9)=\frac{1}{8}n^3(n+1)(n^5-n^4+n^3+3n^2-2n+2)$. This is sequence A217331 in the OEIS.]

31. Consider a group action of $S_2=\{e,t\}$ on the set of entries of a four-by-four matrix. Here e leaves matrices fixed and t maps a matrix to its transpose. Find the cycle index for this group of permutations on the set of sixteen entries.
[Answer: $\frac{1}{2}(a_2^6a_1^4+a_1^{16})$.]

32. Use this to find the number of matrices over the set $M_{4\times 4}(2)$ of four-by-four matrices over $\mathbb{Z}_2$, up to taking the transpose.
[Answer: $\frac{1}{2}(2^{10}+2^{16})=33280$. Thus there are 33280 possible matrices up to this equivalence.]

33. Use this to find the number of matrices over the set $M_{4\times 4}(n)$ of four-by-four matrices over $\mathbb{Z}_n$, up to taking the transpose.
[Answer: $\frac{1}{2}(n^{10}+n^{16})=\frac{1}{2}n^{10}(n^2+1)(n^4-n^2+1)$. This is sequence A170798 in the OEIS.]

34. Consider a group action of S_4 on the set of entries of a four-by-four matrix acting by permuting the rows. Find the cycle index for this group of permutations on the set of sixteen entries.
[Answer: $\frac{1}{24}(6a_2^4a_1^8 + 8a_3^4a_1^4 + 3a_2^8 + 6a_4^4 + a_1^{16})$.]

35. Use this to find the number of matrices over the set $M_{4\times4}(2)$ of four-by-four matrices over $\mathbb{Z}_2$, up to the action of permuting the rows.
[Answer: $\frac{1}{24}(6 \times 2^{12} + 11 \times 2^8 + 6 \times 2^4 + 2^{16}) = 3876$. Thus there are 3876 distinct matrices under this equivalence.]

36. Use this to find the number of matrices over the set $M_{4\times4}(n)$ of four-by-four matrices over $\mathbb{Z}_n$, up to the action of permuting the rows.
[Answer: $\frac{1}{24}(6 \times n^{12} + 11 \times n^8 + 6 \times n^4 + n^{16}) = \frac{1}{24}n^4(n^4+1)(n^4+2)(n^4+3)$.]

37. Consider a group action of $\mathbb{Z}_4 = \{e, r, r^2, r^3\}$ on the set of entries of a four-by-four matrix. Here e leaves matrices fixed and r rotates the matrix ninety degrees, r^2 rotates twice, and r^3 does this three times. Find the cycle index for this group of permutations on the set of sixteen entries.
[Answer: $\frac{1}{4}(2a_4^4 + a_2^8 + a_1^{16})$.]

38. Use this to find the number of matrices over the set $M_{4\times4}(2)$ of four-by-four matrices over $\mathbb{Z}_2$, up to rotations.
[Answer: $\frac{1}{4}(2 \times 2^4 + 2^8 + 2^{16}) = 16456$. Thus there are 16456 distinct matrices up to rotation.]

39. Use this to find the number of matrices over the set $M_{4\times4}(n)$ of four-by-four matrices over $\mathbb{Z}_n$, up to rotations.
[Answer: $\frac{1}{4}(2 \times n^4 + n^8 + n^{16}) = \frac{1}{4}n^4(n^4+1)(n^8-n^4+2)$.]

40. Consider a group action of $V = \{e, h, v, hv\}$ on the set of entries of a four-by-four matrix. Here e leaves matrices fixed and h reflects them across a horizontal line of symmetry, v reflects them across a vertical line of symmetry, and hv does both. Find the cycle index for this group of permutations on the set of sixteen entries.
[Answer: $\frac{1}{4}(3a_2^8 + a_1^{16})$.]

41. Use this to find the number of matrices over the set $M_{4\times4}(2)$ of four-by-four matrices over $\mathbb{Z}_2$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(3 \times 2^8 + 2^{16}) = 16576$. Thus there are 16576 distinct matrices up to our reflections.]

42. Use this to find the number of matrices over the set $M_{4\times4}(n)$ of four-by-four matrices over $\mathbb{Z}_n$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(3 \times n^8 + n^{16}) = \frac{1}{4}n^8(n^8+3)$.]

43. Consider a group action of D_4 on the set of entries of a four-by-four matrix, acting upon them by rotations and reflections. Find the cycle index for this group of permutations on the set of sixteen entries.
[Answer: $\frac{1}{8}(2a_4^4 + 3a_2^8 + 2a_2^6a_1^4 + a_1^{16})$.]

44. Use this to find the number of matrices over the set $M_{4\times4}(2)$ of four-by-four matrices over $\mathbb{Z}_2$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2 \times 2^3 + 2^5 + 4 \times 2^6 + 2^{16}) = 68384$. Thus there are 68384 distinct matrices up to rotations and reflections.]

45. Use this to find the number of matrices over the set $M_{4\times 4}(n)$ of four-by-four matrices over $\mathbb{Z}_n$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2\times n^4+3\times n^8+2\times n^{10}+n^{16}) = \frac{1}{8}n^4(n^{12}+2n^6+3n^4+2)$. This is sequence A217338 in the OEIS.]

46. Consider a group action of $S_2=\{e,t\}$ on the set of entries of a five-by-five matrix. Here e leaves matrices fixed and t maps a matrix to its transpose. Find the cycle index for this group of permutations on the set of twenty-five entries.
[Answer: $\frac{1}{2}(a_2^{10}a_1^5+a_1^{25})$.]

47. Use this to find the number of matrices over the set $M_{5\times 5}(2)$ of five-by-five matrices over $\mathbb{Z}_2$, up to taking the transpose.
[Answer: $\frac{1}{2}(2^{15}+2^{25})=16793600$. Thus there are 33280 possible matrices up to this equivalence.]

48. Use this to find the number of matrices over the set $M_{5\times 5}(n)$ of five-by-five matrices over $\mathbb{Z}_n$, up to taking the transpose.
[Answer: $\frac{1}{2}(n^{10}+n^{25})=\frac{1}{2}n^{15}(n^2+1)(n^8-n^6+n^4-n^2+1)$.]

49. Consider a group action of S_5 on the set of entries of a five-by-five matrix acting by permuting the rows. Find the cycle index for this group of permutations on the set of twenty-five entries.
[Answer: $\frac{1}{120}(10a_2^5a_1^{15}+20a_3^5a_1^{10}+15a_2^{10}a_1^5+30a_4^5a_1^5+20a_2^5a_3^5+24a_5^5+a_1^{25})$.]

50. Use this to find the number of matrices over the set $M_{5\times 5}(2)$ of five-by-five matrices over $\mathbb{Z}_2$, up to the action of permuting the rows.
[Answer: $\frac{1}{120}(10\times 2^{20}+35\times 2^{15}+50\times 2^{10}+24\times 2^5+2^{25})=376992$. Thus there are 376992 distinct matrices under this equivalence.]

51. Use this to find the number of matrices over the set $M_{5\times 5}(n)$ of five-by-five matrices over $\mathbb{Z}_n$, up to the action of permuting the rows.
[Answer: $\frac{1}{120}(10\times n^{20}+35\times n^{15}+50\times n^{10}+24\times n^5+n^{25}) = \frac{1}{120}n^5(n+1)(n^{19}-n^{18}+n^{17}-n^{16}+n^{15}+9n^{14}-9n^{13}+9n^{12}-9n^{11}+9n^{10}+26n^9-26n^8+26n^7-26n^6+26n^5+24n^4-24n^3+24n^2-24n+24)$.]

52. Consider a group action of $\mathbb{Z}_4=\{e,r,r^2,r^3\}$ on the set of entries of a five-by-five matrix. Here e leaves matrices fixed and r rotates the matrix ninety degrees, r^2 rotates twice, and r^3 does this three times. Find the cycle index for this group of permutations on the set of twenty-five entries.
[Answer: $\frac{1}{4}(2a_4^6a_1+a_2^{12}a_1+a_1^{25})$.]

53. Use this to find the number of matrices over the set $M_{5\times 5}(2)$ of five-by-five matrices over $\mathbb{Z}_2$, up to rotations.
[Answer: $\frac{1}{4}(2\times 2^7+2^{13}+2^{25})=8390720$. Thus there are 8390720 distinct matrices up to rotation.]

54. Use this to find the number of matrices over the set $M_{5\times 5}(n)$ of five-by-five matrices over $\mathbb{Z}_n$, up to rotations.
[Answer: $\frac{1}{4}(2\times n^7+n^{13}+n^{25})=\frac{1}{4}n^7(n^2+1)(n^4-n^2+1)(n^{12}-n^6+2)$.]

55. Consider a group action of $V=\{e,h,v,hv\}$ on the set of entries of a five-by-five matrix. Here e leaves matrices fixed and h reflects them across a horizontal line of symmetry, v reflects them across a vertical line of symmetry, and hv does both. Find the cycle index for this group of permutations on the set of twenty-five entries.
[Answer: $\frac{1}{4}(2a_2^{10}a_1^5+a_2^{12}a_1+a_1^{25})$.]

56. Use this to find the number of matrices over the set $M_{5\times 5}(2)$ of five-by-five matrices over $\mathbb{Z}_2$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(2\times 2^{15}+2^{13}+2^{25}) = 16576$. Thus there are 16576 distinct matrices up to our reflections.]

57. Use this to find the number of matrices over the set $M_{5\times 5}(n)$ of five-by-five matrices over $\mathbb{Z}_n$, up to the action of vertical and horizontal reflections.
[Answer: $\frac{1}{4}(2\times n^{15}+n^{13}+n^{25}) = \frac{1}{4}n^13(n^2+1)(n^{10}-n^8+n^6-n^4+n^2+1)$.]

58. Consider a group action of D_4 on the set of entries of a five-by-five matrix, acting upon them by rotations and reflections. Find the cycle index for this group of permutations on the set of twenty-five entries.
[Answer: $\frac{1}{8}(2a_4^6a_1+a_2^{12}a_1+4a_2^{10}a_1^5+a_1^{25})$.]

59. Use this to find the number of matrices over the set $M_{5\times 5}(2)$ of five-by-five matrices over $\mathbb{Z}_2$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2\times 2^7+2^{13}+4\times 2^{15}+2^{25}) = 4211744$. Thus there are 4211744 distinct matrices up to rotations and reflections.]

60. Use this to find the number of matrices over the set $M_{5\times 5}(n)$ of five-by-five matrices over $\mathbb{Z}_n$, up to rotations and reflections.
[Answer: $\frac{1}{8}(2\times n^7+n^{13}+4\times n^{15}+n^{25}) = \frac{1}{8}n^7(n^{18}+4n^8+n^6+2)$.]

Chapter 5

Irreducible Polynomials Over Finite Fields

The number of monic irreducible polynomials of degree n over $\mathbb{Z}_p$ is given by

$$\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})p^d.$$

The product of these monic irreducibles is

$$\prod_{d|n}(x^{p^d}-x)^{\mu(\frac{n}{d})}.$$

You may find the following rules helpful

$$\frac{x^{ab}-1}{x^b-1}=\sum_{i=0}^{i=a-1}x^{bi}=x^{ab-b}+x^{ab-2b}+\cdots+x^{2b}+x^b+1.$$

$$x^{3a}+1=x^{3a}+x^{2a}-x^{2a}-x^a+x^a+1=(x^{2a}-x^a+1)(x^a+1)$$

$$x^{4a}+x^{2a}+1=x^{4a}+x^{3a}+x^{2a}-x^{3a}-x^{2a}-x^a+x^{2a}+x^a+1=(x^{2a}+x^a+1)(x^{2a}-x^a+1)$$

$$x^4+c^2x^2+c^4=(x^2+cx+c^2)(x^2-cx+c^2)$$

$$x^4+(2c-c^2)x^2+c^2=(x^2-cx+c)(x^2+cx+c)$$

$$x^4+(2c-c^4)x^2+c^2=(x^2-c^2x+c)(x^2+c^2x+c)$$

Recall that the following methods can help us find irreducible factors in $\mathbb{Z}_p[x]$ of degree n. Often multiple methods here will work.

- Use the formula to find the number and product of irreducible factors, then factor.
- Write $p(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_3x^3+a_2x^2+a_1x+a_0$ and us the fact that an irreducible must have $p(x)\neq 0$ for all $x\in\mathbb{Z}_p$. This condition will find all irreducible quadratics and cubics, as any reducible must have a linear factor in those cases. For higher degree polynomials this can still help us limit the number of candidates.

- Multiply out all products of polynomials that lead to polynomials of degree n and eliminate these from the list of all possible polynomials.
- Using the quadratic equation, rule out all polynomials for which the discriminant is not a square. [Only for quadratics and only when $p \neq 2$.]
- Use the fact that in $\mathbb{Z}_2$ any irreducible must have an odd number of terms to help limit the number of candidates. [Only when $p = 2$.]

5.1 Finding Irreducible Polynomials Over $\mathbb{Z}_2$

1. How many monic quadratic irreducibles are there over $\mathbb{Z}_2$[1]?
[Answer:
$$\frac{1}{2}\sum_{d|2}\mu(\frac{2}{d})2^d = \frac{1}{2}\left(\mu(2)2^1 + \mu(1)2^2\right) = \frac{1}{2}\left(-2+4\right) = 1.]$$

2. Find the product of these irreducibles.
[Answer:
$$\prod_{d|2}(x^{2^d}-x)^{\mu(\frac{2}{d})} = (x^{2^1}-x)^{\mu(\frac{2}{1})}(x^{2^2}-x)^{\mu(\frac{2}{2})} = (x^2-x)^{-1}(x^4-x) = \frac{x(x^3-1)}{x(x-1)} = x^2+x+1$$
Notice that since there is only one, the product formula immediately gives us the only monic irreducible quadratic.]

3. Use the fact that any polynomial over $\mathbb{Z}_2$ with an even number of terms has one as a root, and hence $(x-1)$ as a linear factor, to find all irreducible quadratics.
[Answer: A quadratic with one or three terms must be x^2 or x^2+x+1. As x^2 is $x \cdot x$ this leaves x^2+x+1 as our only irreducible.]

4. How many monic cubic irreducibles are there over $\mathbb{Z}_2$?
[Answer:
$$\frac{1}{3}\sum_{d|3}\mu(\frac{3}{d})2^d = \frac{1}{3}\left(\mu(3)2^1 + \mu(1)2^3\right) = \frac{1}{3}\left(-2+8\right) = \frac{6}{3} = 2.]$$

5. Find the product of these irreducibles.
[Answer:
$$\prod_{d|3}(x^{2^d}-x)^{\mu(\frac{3}{d})} = (x^{2^1}-x)^{\mu(\frac{3}{1})}(x^{2^3}-x)^{\mu(\frac{3}{3})} = (x^2-x)^{-1}(x^8-x) =$$
$$\frac{x(x^7-1)}{x(x-1)} = \frac{x^7-1}{x-1} = x^6+x^5+x^4+x^3+x^2+x+1.$$
Notice that since there is only one, the product formula immediately gives us the only monic irreducible quadratic.]

[1] We actually can drop the "monic" from this section, because in $\mathbb{Z}_2$, there is only one non-zero possibility.

6. Use the fact that any irreducible polynomial over $\mathbb{Z}_2$ must have an odd number of terms to find all irreducible cubics.
[Answer: Any irreducible must have one or three terms. If it has one term it must be $x^3 = x \cdot x \cdot x$ and hence reducible, thus it must have three terms. One of those terms must be the constant term, for without that, zero would be a root. The only possibilities left with three terms are $x^3 + x^2 + 1$ and $x^3 + x + 1$.]

7. To find the monic cubic irreducible polynomials consider the set of polynomials of the form $p(x) = x^3 + ax^2 + bx + 1$. None of these can have x as a factor due to the nonzero constant term. As any reducible cubic must have a linear factor, if $p(1) \neq 0$ we know our polynomial is irreducible. Use this to find all irreducible cubics in $\mathbb{Z}_2[x]$.
[Answer: The irreducibles are the polynomials where $p(1) = 1 + a + b + 1 = 1$, and thus those where $a + b = 1$. Thus $a = 1$ and $b = 0$ or $a = 0$ and $b = 1$. We therefore have $x^3 + x + 1$ and $x^3 + x^2 + 1$ as our two irreducibles. We can check our work as we know the product of these two must be $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.]

8. We can find these irreducible polynomials in other ways as well. Consider that an irreducible cubic must either be the product of an irreducible quadratic with a linear term, or a product of three linear terms. List all polynomials you can arrive at through such products.
[Answer:

$$\begin{array}{ccc} (x^2+x+1)(x-1) = x^3+1, & (x^2+x+1)x = x^3+x^2+x, & (x-1)(x-1)(x-1) = x^3+x^2+x+1, \\ (x-1)(x-1)x = x^3+x, & (x-1)(x)(x) = x^3+x^2, & (x)(x)(x) = x^3. \end{array}$$

We take the single irreducible quadratic times both possible linear factors, and then all possible products of three linear factors.]

9. List all cubic polynomials in $\mathbb{Z}_2[x]$.
[Answer:

$$\begin{array}{cccc} x^3 & x^3+1, & x^3+x, & x^3+x+1, \\ x^3+x^2, & x^3+x^2+1, & x^3+x^2+x, & x^3+x^2+x+1. \end{array}$$

We organize them by coefficients.]

10. List all irreducible cubic polynomials in $\mathbb{Z}_2[x]$.
[Answer:

$$\begin{array}{cccc} x^3, & x^3+1, & x^3+x, & \mathbf{x^3+x+1}, \\ x^3+x^2, & \mathbf{x^3+x^2+1}, & x^3+x^2+x, & x^3+x^2+x+1. \end{array}$$

We remove the polynomials from our previous list and arrive at two possibilities. We can also remove all polynomials with an even number of terms, and polynomials with zero constant term. In this case, this is enough to give us all irreducibles.]

11. How many monic quartic irreducibles are there over $\mathbb{Z}_2$?
[Answer:
$$\frac{1}{4}\sum_{d|4}\mu(\frac{4}{d})2^d = \frac{1}{4}\left(\mu(4)2^1+\mu(2)2^2+\mu(1)2^4\right) = \frac{1}{4}(-4+16) = \frac{12}{4} = 3.]$$

12. Find the product of these irreducibles.
[Answer:
$$\prod_{d|4}(x^{2^d}-x)^{\mu(\frac{4}{d})} = (x^{2^1}-x)^{\mu(\frac{4}{1})}(x^{2^2}-x)^{\mu(\frac{4}{2})}(x^{2^4}-x)^{\mu(\frac{4}{4})} = (x^4-x)^{-1}(x^{16}-x)^1 =$$
$$\frac{(x^{16}-x)}{(x^4-x)} = \frac{x(x^{15}-1)}{x(x^3-1)} = \frac{x^{15}-1}{x^3-1} = x^{12}+x^9+x^6+x^3+1.]$$

13. Use the fact that any irreducible polynomial over $\mathbb{Z}_2$ must have an odd number of terms to find all irreducible quartics.
[Answer: We also must include a constant term. This gives us only these possibilities:
$$x^4+x^3+x^2+x+1, x^4+x+1, x^4+x^3+1, x^4+x^2+1.$$
Note that these can not all be irreducible, as there are exactly three irreducible quartics. We do know that none of these have linear factors, so the only way for one not to be irreducible, is if it is a product of two quadratic irreducibles. There is only one irreducible quadratic and $(x^2+x+1)(x^2+x+1) = x^4+x^2+1$. We thus know the other three are our irreducibles, so our list is
$$x^4+x^3+x^2+x+1, x^4+x+1, x^4+x^3+1.]$$

14. How many monic quintic irreducibles are there over $\mathbb{Z}_2$?
[Answer:
$$\frac{1}{5}\sum_{d|5}\mu(\frac{5}{d})2^d = \frac{1}{5}\left(\mu(5)2^1+\mu(5)2^5\right) = \frac{1}{5}(-2+32) = \frac{30}{5} = 6.]$$

15. Find the product of these irreducibles.
[Answer:
$$\prod_{d|5}(x^{2^d}-x)^{\mu(\frac{5}{d})} = (x^{2^1}-x)^{\mu(\frac{5}{1})}(x^{2^5}-x)^{\mu(\frac{5}{5})} = (x^2-x)^{-1}(x^{32}-x)^1 =$$
$$\frac{(x^{32}-x)}{(x^2-x)} = \frac{x(x^{31}-1)}{x(x-1)} = \frac{x^{31}-1}{x-1} = x^{30}+x^{29}+\cdots+x^2+x+1.]$$

16. Use the fact that any irreducible polynomial over $\mathbb{Z}_2$ must have an odd number of terms to find all irreducible quintics.
[Answer: We start by listing polynomials avoiding linear factors. These all have an odd number of terms and a non-zero constant term. Our eight possibilities are
$$\begin{array}{cccc} x^5+x^4+x^3+x^2+1, & x^5+x^4+x^3+x+1, & x^5+x^4+x^2+x+1, & x^5+x^3+x^2+x+1, \\ x^5+x^4+1, & x^5+x^3+1, & x^5+x^2+1, & x^5+x+1. \end{array}$$

Of these, we must remove those that are a product of two irreducibles. This means removing products of an irreducible quadratic and cubic. The only possibilities are $(x^2+x+1)(x^3+x+1) = x^5+x^4+1$ and $(x^2+x+1)(x^3+x^2+1) = x^5+x+1$. After removing these two, we get our final list of six irreducibles:

$$\begin{array}{ccc} x^5+x^4+x^3+x^2+1, & x^5+x^4+x^3+x+1, & x^5+x^4+x^2+x+1, \\ x^5+x^3+x^2+x+1, & x^5+x^3+1, & x^5+x^2+1.] \end{array}$$

17. How many monic sextic irreducibles are there over $\mathbb{Z}_2$?
[Answer:

$$\frac{1}{6}\sum_{d|6}\mu(\frac{6}{d})2^d = \frac{1}{6}\left(\mu(6)2^1+\mu(3)2^2+\mu(2)2^3+\mu(1)2^6\right) = \frac{1}{6}(2-4-8+64) = \frac{54}{6} = 9.]$$

18. Find the product of these irreducibles. You may leave your answer as a rational function.
[Answer:

$$\prod_{d|6}(x^{2^d}-x)^{\mu(\frac{6}{d})} = (x^{2^1}-x)^{\mu(\frac{6}{1})}(x^{2^2}-x)^{\mu(\frac{6}{2})}(x^{2^3}-x)^{\mu(\frac{6}{3})}(x^{2^6}-x)^{\mu(\frac{6}{6})} =$$

$$(x^2-x)^1(x^4-x)^{-1}(x^8-x)^{-1}(x^{64}-x)^1 = \frac{(x^2-x)(x^{64}-x)}{(x^4-x)(x^8-x)} = \frac{(x-1)(x^{63}-1)}{(x^3-1)(x^7-1)}$$

This may not be easy to simplify.

19. Use the fact that any irreducible polynomial over $\mathbb{Z}_2$ must have an odd number of terms to find all irreducible sextics.
[Answer: We first make a list of sextics with an odd number of terms and non-zero constant. This gives us:

$$\begin{array}{cccc} x^6+x^5+x^4+x^3+x^2+x+1, & x^6+x^3+x^2+x+1, & x^6+x^4+x^2+x+1, & x^6+x^4+x^3+x+1, \\ x^6+x^4+x^3+x^2+1, & x^6+x^5+x^2+x+1, & x^6+x^5+x^3+x+1, & x^6+x^5+x^3+x^2+1, \\ x^6+x^5+x^4+x+1, & x^6+x^5+x^4+x^2+1, & x^6+x^5+x^4+x^3+1, & x^6+x+1, \\ x^6+x^2+1, & x^6+x^3+1, & x^6+x^4+1, & x^6+x^5+1. \end{array}$$

We have to remove products of two irreducible cubics, products of irreducible quadratics with irreducible quartics, and products of three irreducible quadratics. These are

$$\begin{array}{ccc} (x^4+x^3+x^2+x+1)(x^2+x+1) & = & x^6+x^4+x^3+x^2+1, \\ (x^4+x+1)(x^2+x+1) & = & x^6+x^5+x^4+x^3+1, \\ (x^4+x^3+1)(x^2+x+1) & = & x^6+x^3+x^2+x+1, \\ (x^3+x+1)(x^3+x^2+1) & = & x^6+x^5+x^4+x^3+x^2+x+1, \\ (x^3+x+1)(x^3+x+1) & = & x^6+x^2+1, \\ (x^3+x^2+1)(x^3+x^2+1) & = & x^6+x^4+1, \\ (x^2+x+1)(x^2+x+1)(x^2+x+1) & = & x^6+x^5+x^3+x+1. \end{array}$$

This leaves us with

$$\begin{array}{ccc} x^6+x^4+x^2+x+1, & x^6+x^4+x^3+x+1, & x^6+x^5+x^2+x+1, \\ x^6+x^5+x^3+x^2+1, & x^6+x^5+x^4+x+1, & x^6+x^5+x^4+x^2+1, \\ x^6+x+1, & x^6+x^3+1, & x^6+x^5+1.] \end{array}$$

5.2 Finding Irreducible Polynomials Over $\mathbb{Z}_3$

1. How many monic quadratic irreducibles are there over $\mathbb{Z}_3$?
[Answer:
$$\frac{1}{2}\sum_{d|2}\mu(\frac{2}{d})3^d = \frac{1}{2}\left(\mu(2)3^1+\mu(1)3^2\right) = \frac{1}{2}(-3+9) = 3.]$$

2. Find the product of these irreducibles.
[Answer:
$$\prod_{d|2}(x^{3^d}-x)^{\mu(\frac{2}{d})} = (x^{3^1}-x)^{\mu(\frac{2}{1})}(x^{3^2}-x)^{\mu(\frac{2}{2})} = (x^3-x)^{-1}(x^9-x) =$$
$$\frac{x(x^8-1)}{x(x^2-1)} = \frac{x^8-1}{x^2-1} = x^6+x^4+x^2+1.]$$

3. Factor $x^6+x^4+x^2+1$ by grouping the four terms into two even groups.
[Answer:
$$x^6+x^4+x^2+1 = x^4(x^2+1)+x^2+1 = (x^2+1)(x^4+1).]$$

4. We know x^4+1 can not have linear factors. Find the two quadratic factors of x^4+1 by setting it equal to the product $(x^2+ax+c)(x^2+bx+d)$ and solving for a, b, c, and d.
[Answer: We get that a, b, c and d must satisfy
$$\begin{array}{c} a+b=0 \\ ab+c+d=0 \\ cb+ad=0 \\ cd=1 \end{array}.$$
We immediately get that $a=-b$ so our remaining equations become
$$\begin{array}{c} -b^2+c+d=0 \\ b(c-d)=0 \\ cd=1. \end{array}$$
We know that either b or $c-d$ must be zero. If b is zero, $c+d=0$ so $d=-c$, but then we need to solve $-c^2=1$ which is impossible as two is not a square in $\mathbb{Z}_3$. Thus $c-d=0$ and $d=c$. We have that $c^2=1$ so c is either 1 or 2. If $c=1$ then $d=1$ and $-b^2+c+d=0$ implies $2=b^2$ which is impossible. Thus both c and d are two. Now we have $b^2=1$ so b can be one or two, but as a is negative b this gives us the quadratics x^2+x-1 and x^2-x-1 in some order.]

5. Use a formula to rewrite x^4+1 as a product of two factors.
[Answer: We can use the formula given for $x^4+(2c-c^2)x^2+c^2$ in the case $c=-1$ here as then $2c-c^2=-2-1=0$, so this will give us the factorization of x^4+1. The formula gives
$$x^4+1 = x^4-3x^2+1 = (x^2+x-1)(x^2-x-1).]$$

6. Use the results of our factorizations to list the three irreducible monic quadratics over $\mathbb{Z}_3$.
[Answer:
$$x^2+1, x^2+x-1, x^2-x-1.]$$

7. We can also find these three another way as an irreducible quadratic or cubic must have a root. A monic polynomial of degree two must have the form $p(x) = x^2 + bx + c$. If $c = 0$ then zero would be a root, thus we have two cases for each of three possible values for b. For $b = 0$, find which values of c give an irreducible polynomial by taking $p(x)$ for $x \in \mathbb{Z}_3$. For reducible polynomials, list all roots and factor.
[Answer: $x^2 + 1$ is irreducible. $x^2 + 2 = x^2 - 3x + 2 = (x - 1)(x - 2)$ which has 1 and 2 as roots.]

8. Do the same for $p(x) = x^2 + bx + c$ in the $b = 1$ case.
[Answer: $x^2 + x + 2$ is irreducible. $x^2 + x + 1 = x^2 - 2x + 1 = (x - 1)^2$ which has only 1 as a root.]

9. Do the same for $p(x) = x^2 + bx + c$ in the $b = 2$ case.
[Answer: $x^2 + 2x + 2$ is irreducible. $x^2 + 2x + 1 = x^2 - 4x + 4 = (x - 2)^2$ which has only 2 as a root.]

10. Use the results of this brute force method to list the three irreducible monic quadratics over $\mathbb{Z}_3$.
[Answer:
$$x^2+1, x^2+x-1, x^2-x-1.]$$

11. Instead of examining the outputs of $p(x)$ for $x \in \mathbb{Z}_3$ to see when $p(x)$ equals zero, we can approach things differently by taking all products of two monic linear factors[2]. None of the resulting polynomoials will be irreducible. Use this to find all irreducible monic quadratic polynomials.
[Answer: We know the following polynomials are reducible:
$$\begin{array}{ccc} (x)(x) = x^2, & (x-1)(x-1) = x^2+x+1, & (x-2)(x-2) = x^2+2x+1, \\ (x-1)x = x^2+2x, & (x-2)x = x^2+x, & (x-1)(x-2) = x^2+2. \end{array}$$
This leaves only three possibilities left for irreducibles:
$$x^2+1, x^2+x+2, x^2+2x+2.]$$

12. We can also find the irreducible quadratics here in another way, by using the quadratic equation. If the discriminant of $b^2 - 4ac$ happens to be a square, then the quadratic will have a root, and hence be reducible. since we are restricting ourselves to monic polynomials, we are looking to find and eliminate all b and c so that $b^2 - 4c = b^2 - c$ equals 0 or 1, as these are the only squares in $\mathbb{Z}_3$. Use this method to find all irreducible monic quadratics in $\mathbb{Z}_3[x]$.
[Answer: To see when $b^2 - c = 0$ or $b^2 - c = 1$, we can plug in each possible value of b. There will be a unique c which satisfies it equal to b^2 or $b^2 - 1$ depending on which equation we use.

b	$c = b^2$	$c = b^2 - 1$
0	0	2
1	1	0
2	1	0

[2]It suffices to examine monic linear factors, because for $(x - a)(x - b) = (2x - 2a)(2x - 2b) = (2x + a)(2x + b)$. so we can travel back and forth between the monic and non-monic products.

This tells us that the reducibles are

$$\begin{array}{cc} x^2 & x^2+2 \\ x^2+x+1 & x^2+x \\ x^2+2x+1 & x^2+2x \end{array}$$

which leaves

$$x^2+1, x^2+x+2, x^2+2x+2$$

as our irreducibles.]

13. How many monic cubic irreducibles are there over $\mathbb{Z}_3$?
[Answer:
$$\frac{1}{3}\sum_{d|3}\mu(\frac{3}{d})3^d = \frac{1}{3}\left(\mu(3)3^1+\mu(1)3^3\right) = \frac{1}{3}(-3+27) = \frac{24}{3} = 8.]$$

14. Find the product of these irreducibles.
[Answer:
$$\prod_{d|3}(x^{3^d}-x)^{\mu(\frac{3}{d})} = (x^{3^1}-x)^{\mu(\frac{3}{1})}(x^{3^3}-x)^{\mu(\frac{3}{3})} = (x^3-x)^{-1}(x^{27}-x) =$$
$$\frac{x(x^{26}-1)}{x(x^2-1)} = \frac{x^{26}-1}{x^2-1} = x^{24}+x^{22}+\cdots+x^2+1.]$$

15. Use the fact that a reducible monic cubic polynomial in $\mathbb{Z}_3[x]$ must have a root to find the eight irreducible polynomials. Such polynomials must have the form $p(x) = x^3+ax^2+bx+c$ with $c \neq 0$. Examine the six polynomials in the $a=0$ case and state which polynomials are irreducible.
[Answer: x^3+bx+c is reducible when $1+b+c=0$ or $-1-b+c=0$. The first occurs for $b=1, c=1$ and $b=0, c=2$ and the second occurs for $b=1, c=2$ or $b=0, c=1$. These create four distinct reducibles. This leaves $b=2, c=1$ and $b=2, c=2$ forming our two irreducible cases of x^3+2x+1 and x^3+2x+2.]

16. Examine the six polynomials in the $a=1$ case and state which polynomials are irreducible.
[Answer: x^3+x^2+bx+c is reducible when $2+b+c=0$ or $-b+c=0$. The first occurs for $b=0, c=1$, and $b=2, c=2$, and the second occurs for $b=1, c=1$, and $b=2, c=2$. Due to repetition, these create three distinct reducibles. This leaves $b=0, c=2$, $b=1, c=2$ and $b=2, c=1$ forming our three irreducible cases of x^3+x^2+2, x^3+x^2+x+2 and x^3+x^2+2x+1.]

17. Examine the six polynomials in the $a=2$ case and state which polynomials are irreducible.
[Answer: x^3-x^2+bx+c is reducible when $b+c=0$ or $1-b+c=0$. The first occurs for $b=1, c=2$, and $b=2, c=1$, and the second occurs for $b=0, c=2$, and $b=2, c=1$. Due to repetition, these create three distinct reducibles. This leaves $b=0, c=1$, $b=1, c=1$ and $b=2, c=2$ forming our three irreducible cases of x^3+2x^2+1, x^3+2x^2+x+1 and x^3+2x^2+2x+2.]

18. Use what we've compiled to list all irreducible monic cubic polynomials over $\mathbb{Z}_3$.
[Answer:
$$\begin{array}{cccc} x^3+2x+1, & x^3+2x+2, & x^3+x^2+2, & x^3+x^2+x+2, \\ x^3+x^2+2x+1, & x^3+2x^2+1, & x^3+2x^2+x+1, & x^3+2x^2+2x+2.] \end{array}$$

19. We can arrive at this list in another way: by finding all the reducible monic polynomials. Each of these must be the product of an irreducible monic with a degree one monic polynomial, or a product of three monic degree one polynomials [3]. Find all monic cubics in $\mathbb{Z}_3[x]$ which are the product of an irreducible quadratic and a linear factor.
[Answer: Our three irreducible quadratics are x^2+1, x^2+x-1, and x^2-x-1. We multiply each of these in turn by $x, x+1$ and $x+2$.

$$\begin{array}{ccc}
(x^2+1)x & = & x^3+x, \\
(x^2+x-1)x & = & x^3+x^2+2x, \\
(x^2-x-1)x & = & x^3+2x^2+2x, \\
(x^2+1)(x+1) & = & x^3+x^2+x+1, \\
(x^2+x-1)(x+1) & = & x^3+2x^2+2, \\
(x^2-x-1)(x+1) & = & x^3+x+2, \\
(x^2+1)(x+2) & = & x^3+2x^2+x+2, \\
(x^2+x-1)(x+2) & = & x^3+x+1, \\
(x^2-x-1)(x+2) & = & x^3+x^2+1.
\end{array}$$

This allows us to find the nine cubics for our answer.]

20. Next find the monic cubics arising from three linear factors by considering the one, two, and three distinct root cases.

$$\begin{array}{ccc}
x^3 & = & x^3, \\
(x-1)^3 & = & x^3+2, \\
(x-2)^3 & = & x^3+1, \\
x^2(x-1) & = & x^3+2x^2, \\
(x-1)^2(x) & = & x^3+x^2+x, \\
(x-2)^2x & = & x^3+2x^2+x, \\
x^2(x-2) & = & x^3+x^2, \\
(x-1)^2(x-2) & = & x^3+2x^2+2x+1, \\
(x-2)^2(x-1) & = & x^3+x^2+2x+2, \\
(x-2)(x-1)x & = & x^3+2x.
\end{array}$$

21. List all monic cubics over $\mathbb{Z}_3$.
[Answer:

---

[3]You may worry for a moment about the possibility of non monic polynomials whose product is monic. This can indeed happen, but it's fine if it does. In order for a product to be monic, we need the number of polynomials with leading term two to be even. In this case, they can be converted by multiplying both by two. As two times two is one there, this is yet another example where the trick is multiplication by one. For example $(2x+1)(2x+2)$ is the same as $2\cdot 2\cdot(2x+1)(2x+2) = (2(2x+1))(2(2x+2)) = (x+2)(x+1)$

$$\begin{array}{ccc}
x^3, & x^3+1, & x^3+2, \\
x^3+x, & x^3+x+1, & x^3+x+2, \\
x^3+2x, & x^3+2x+1, & x^3+2x+2, \\
x^3+x^2, & x^3+x^2+1, & x^3+x^2+2, \\
x^3+x^2+x, & x^3+x^2+x+1, & x^3+x^2+x+2, \\
x^3+x^2+2x, & x^3+x^2+2x+1, & x^3+x^2+2x+2, \\
x^3+2x^2, & x^3+2x^2+1, & x^3+2x^2+2, \\
x^3+2x^2+x, & x^3+2x^2+x+1, & x^3+2x^2+x+2, \\
x^3+2x^2+2x, & x^3+2x^2+2x+1, & x^3+2x^2+2x+2.
\end{array}$$

We organize them into three columns based on last coefficient, and write all possibilities.]

22. Use the results of our computations to list all the irreducible monic cubics over $\mathbb{Z}_3$.
[Answer:

$$\begin{array}{ccc}
x^3, & x^3+1, & x^3+2, \\
x^3+x, & x^3+x+1, & x^3+x+2, \\
x^3+2x, & \boldsymbol{x^3+2x+1,} & \boldsymbol{x^3+2x+2,} \\
x^3+x^2, & x^3+x^2+1, & \boldsymbol{x^3+x^2+2,} \\
x^3+x^2+x, & x^3+x^2+x+1, & \boldsymbol{x^3+x^2+x+2,} \\
x^3+x^2+2x, & \boldsymbol{x^3+x^2+2x+1,} & x^3+x^2+2x+2, \\
x^3+2x^2, & \boldsymbol{x^3+2x^2+1,} & x^3+2x^2+2, \\
x^3+2x^2+x, & \boldsymbol{x^3+2x^2+x+1,} & x^3+2x^2+x+2, \\
x^3+2x^2+2x, & x^3+2x^2+2x+1, & \boldsymbol{x^3+2x^2+2x+2.}
\end{array}$$

We simply remove each of the reducible cubics that we found before.]

5.3 Finding Irreducible Polynomials of Over $\mathbb{Z}_5$

1. How many monic quadratic irreducibles are there over $\mathbb{Z}_5$?
[Answer:
$$\frac{1}{2}\sum_{d|2}\mu\left(\frac{2}{d}\right)5^d = \frac{1}{2}\left(\mu(2)5^1+\mu(1)5^2\right) = \frac{1}{2}\left(-5+25\right) = 10.]$$

2. Find the product of these irreducibles.
[Answer:
$$\prod_{d|2}(x^{5^d}-x)^{\mu(\frac{2}{d})} = (x^{5^1}-x)^{\mu(\frac{2}{1})}(x^{5^2}-x)^{\mu(\frac{2}{2})} = (x^5-x)^{-1}(x^{25}-x) = \frac{x(x^{24}-1)}{x(x^4-1)} = \frac{x^{24}-1}{x^4-1} =$$
$$x^{20}+x^{16}+x^{12}+x^8+x^4+1.]$$

3. Use the fact that $x^{20}+x^{16}+x^{12}+x^8+x^4+1 = x^{12}(x^8+x^4+1)+x^8+x^4+1$ to rewrite this polynomial as a product of two factors.
[Answer:
$$x^{12}(x^8+x^4+1)+x^8+x^4+1 = (x^8+x^4+1)(x^{12}+1).]$$

4. Use a formula to rewrite $x^{12}+1$ as a product of two factors.
[Answer:
$$x^{12}+x^8-x^8-x^4+x^4+1=(x^4+1)x^8-(x^4+1)x^4+(x^4+1)=(x^4+1)(x^8-x^4+1).]$$

5. Use a formula to rewrite x^8+x^4+1 as a product of two factors.
[Answer:
$$x^8+x^6+x^4-x^6-x^4-x^2+x^4+x^2+1=(x^4-x^2+1)(x^4+x^2+1).]$$

6. Use a formula to rewrite x^4+x^2+1 as a product of two factors.
[Answer:
$$x^4+x^2+1=x^4+x^3+x^2-x^3-x^2-x+x^2+x+1=(x^2-x+1)(x^2+x+1).]$$

7. Combine what we've done to reduce the polynomial $x^{20}+x^{16}+x^{12}+x^8+x^4+1$ to a product of five factors.
[Answer:
$$x^{20}+x^{16}+x^{12}+x^8+x^4+1=(x^8+x^4+1)(x^{12}+1)=$$
$$(x^4-x^2+1)(x^4+x^2+1)(x^4+1)(x^8-x^4+1)=$$
$$(x^4-x^2+1)(x^2-x+1)(x^2+x+1)(x^4+1)(x^8-x^4+1).]$$

8. Use the fact that $x^4+1=x^4-4$ to factor this polynomial in $\mathbb{Z}_5[x]$.
[Answer:
$$x^4+1=x^4-4=(x^2+2)(x^2-2).]$$

9. Use the fact that $x^4+1=x^4+5x^2+6$ to factor this polynomial in $\mathbb{Z}_5[x]$.
[Answer:
$$x^4+1=x^4+5x^2+6=(x^2+2)(x^2+3).]$$

10. Factor x^4-x^2+1 by writing it as a product of two arbitrary monic quadratics.
[Answer: We set
$$x^2-x^2+1=(x^2+ax+c)(x^2+bx+d)$$
to get the system of equations
$$\begin{array}{c} a+b=0 \\ ab+c+d=-1 \\ cb+ad=0 \\ cd=1 \end{array}.$$
We immediately get that $a=-b$ so our remaining equations become
$$\begin{array}{c} -b^2+c+d=-1 \\ b(c-d)=0 \\ cd=1. \end{array}$$

We know either b or $c-d$ must be zero. If $b=0$ then $c+d=-1$ so $d=-1-c$. Plugging this into $cd=1$ tells us $c^2+c+1=0$ which is impossible. We thus conclude $b \neq 0$, so we know c must equal d. As $c^2=1$ we know $c=d=1$ or $c=d=4$. If $c=d=1$ we get $-b^2+1+1=-1$ so $b^2=3$ which is impossible. Therefore $c=d=4$. As $-b^2+c+d=-1$ we have $b^2=4$ which tells us that b is two or three. As $a=-b$, either way we get the factorization

$$x^4-x^2+1=(x^2+2x+4)(x^2+3x+4)$$

and are done.]

11. Use one of our formulas to factor x^4-x^2+1 in $\mathbb{Z}_5[x]$.
[Answer: Notice that our polynomial if equal to $x^4+c^2x^2+c^4$ in the case that $c=-2$. Thus we can plug that c into our formula to get

$$(x^2-cx+c)(x^2+cx+c)=(x^2+2x+4)(x^2-2x+4).]$$

12. Use the result of this last factorization to factor that x^8-x^4+1.
[Answer:Simply plug x^2 into the equation x^4-x^2+1 which we've already factored as $(x^2+2x+4)(x^2-2x+4)$. Plugging x^2 gives us

$$x^8-x^4+1=(x^4+2x^2+4)(x^4-2x^2+4).]$$

13. Factor x^4+2x^2+4 by writing it as a product of two arbitrary monic quadratics.
[Answer: We set

$$x^2+2x^2+4=(x^2+ax+c)(x^2+bx+d)$$

to get the system of equations

$$\begin{gathered} a+b=0 \\ ab+c+d=2 \\ cb+ad=0 \\ cd=4 \end{gathered}.$$

We immediately get that $a=-b$ so our remaining equations become

$$\begin{gathered} -b^2+c+d=2 \\ b(c-d)=0 \\ cd=4. \end{gathered}$$

We know either b or $c-d$ must be zero. If $b=0$ then $c+d=2$ so $d=2-c$. Plugging this into $cd=4$ tells us that $2c-c^2=4$ so $c^2-2c+4=0$ which is impossible. We thus conclude $b \neq 0$, so we know c must equal d. As $c^2=4$ we know $c=d=2$ or $c=d=3$. If $c=d=2$ we get $-b^2+2+2=2$ so $b^2=2$ which is impossible. Therefore $c=d=3$. As $-b^2+c+d=2$ we have $b^2=4$ which tells us that b is two or three. As $a=-b$, either way we get the factorization

$$x^4+2x^2+4=(x^2-3x+3)(x^2+3x+3)$$

and are done.]

14. Use a formula to factor $x^4 + 2x^2 + 4$ in $\mathbb{Z}_5[x]$.
[Answer: Notice that this polynomial is equal to $x^4 + (2c - c^2)x^2 + c^2$ in the case where $c = 3$. Our formula tells us this factors as $(x^2 - cx + c)(x^2 + cx + c)$ which gives us
$$x^4 + 2x^2 + 4 = (x^2 - 3x + 3)(x^2 + 3x + 3).]$$

15. Factor $x^4 - 2x^2 + 4$ by writing it as a product of two arbitrary monic quadratics.
[Answer: We set
$$x^4 - 2x^2 + 4 = (x^2 + ax + c)(x^2 + bx + d)$$
to get the system of equations
$$\begin{aligned} a + b &= 0 \\ ab + c + d &= -2 \\ cb + ad &= 0 \\ cd &= 4 \end{aligned}.$$
We immediately get that $a = -b$ so our remaining equations become
$$\begin{aligned} -b^2 + c + d &= -2 \\ b(c - d) &= 0 \\ cd &= 4. \end{aligned}$$
We know either b or $c - d$ must be zero. If $b = 0$ then $c + d = -2$ so $d = -2 - c$. Plugging this into $c(-2 - c) = 4$ tells us that $-2c - c^2 = 4$ so $c^2 + 2c + 4 = 0$ which is impossible. We thus conclude $b \neq 0$, so we know c must equal d. As $c^2 = 4$ we know $c = d = 2$ or $c = d = 3$. If $c = d = 3$ we get $-b^2 + 3 + 3 = -2$ so $b^2 = 3$ which is impossible. Therefore $c = d = 2$. As $-b^2 + c + d = -2$ we have $b^2 = 1$ which tells us that b is one or four. As $a = -b$, either way we get the factorization
$$x^4 - 2x^2 + 4 = (x^2 - x + 2)(x^2 + x + 2)$$
and are done.]

16. Use a formula to factor $x^4 - 2x^2 + 4$ in $\mathbb{Z}_5[x]$.
[Answer: Notice that when $c = 2$ our polynomial is equal to $x^4 + (2c - c^4)x^2 + c^2$. Our factorization formula then tells us this is equal to $(x^2 - c^2x + c)(x^2 + c^2x + c)$ which gives us
$$x^4 - 2x^2 + 4 = (x^2 - x + 2)(x^2 + x + 2).]$$

17. Use the results of our factorizations to list the ten irreducible monic quadratics over $\mathbb{Z}_5$.
[Answer:
$$x^2 + 2, x^2 + 3, x^2 + x + 1, x^2 + x + 2, x^2 + 2x + 3,$$
$$x^2 + 2x + 4, x^2 + 3x + 3, x^2 + 3x + 4, x^2 + 4x + 1, x^2 + 4x + 2.]$$

18. We can also find these ten monic irreducibles another way as an irreducible quadratic or cubic must have a root. A monic polynomial of degree two must have the form $p(x) = x^2 + bx + c$. If $c = 0$ then zero would be a root, thus we have four cases for each possible b we need to check for roots. For $b = 0$, find which values of c give an irreducible polynomial by taking $p(x)$ for $x \in \mathbb{Z}_5$. For reducible polynomials, list all roots and factor.
[Answer: $x^2 + 2$ and $x^2 + 3$ are irreducible. $x^2 + 1 = x^2 - 5x + 6 = (x - 2)(x - 3)$ has 2 and 3 as roots. $x^2 + 4 = x^2 - 5x + 4 = (x - 1)(x - 4)$ has 1 and 4 as roots.]

19. Do the same for $p(x) = x^2 + bx + c$ in the $b = 1$ case.
[Answer: $x^2 + x + 1$ and $x^2 + x + 2$ are irreducible. $x^2 + x + 3 = x^2 - 4x + 3 = (x - 1)(x - 3)$ has 1 and 3 as roots. $x^2 + x + 4 = x^2 - 4x + 4 = (x - 2)(x - 2)$ has 2 as its only root.]

20. Do the same for $p(x) = x^2 + bx + c$ in the $b = 2$ case.
[Answer: $x^2 + 2x + 3$ and $x^2 + 2x + 4$ are irreducible. $x^2 + 2x + 1 = x^2 - 8x + 16 = (x - 4)(x - 4)$ has 4 as its only root. $x^2 + 2x + 2 = x^2 - 3x + 2 = (x - 1)(x - 2)$ has 1 and 2 as roots.]

21. Do the same for $p(x) = x^2 + bx + c$ in the $b = 3$ case.
[Answer: $x^2 + 3x + 3$ and $x^2 + 3x + 4$ are irreducible. $x^2 + 3x + 1 = x^2 - 2x + 1 = (x - 1)(x - 1)$ has 1 as its only root. $x^2 + 3x + 2 = x^2 - 7x + 12 = (x - 3)(x - 4)$ has 3 and 4 as roots.]

22. Do the same for $p(x) = x^2 + bx + c$ in the $b = 4$ case.
[Answer: $x^2 + 4x + 1$ and $x^2 + 4x + 2$ are irreducible. $x^2 + 4x + 3 = x^2 - 6x + 8 = (x - 2)(x - 4)$ has 2 and 4 as its roots. $x^2 + 4x + 4 = x^2 - 6x + 9 = (x - 3)(x - 3)$ has 3 as its only root.]

23. Use the results of this brute force method to list the ten irreducible monic quadratics over $\mathbb{Z}_5$.
[Answer:
$$x^2 + 2, x^2 + 3, x^2 + x + 1, x^2 + x + 2, x^2 + 2x + 3,$$
$$x^2 + 2x + 4, x^2 + 3x + 3, x^2 + 3x + 4, x^2 + 4x + 1, x^2 + 4x + 2.]$$

24. We can also find these ten by calculating the reducible monics in a different way. Instead of going by the coefficients in $p(x) = x^2 + bx + c$, we can also go through all possible products of two monic factors to find all reducibles [4]. Make a list of all products of two linear monic polynomials in $\mathbb{Z}_5[x]$.
$$\begin{array}{ccc} x(x) = x^2 & x(x+1) = x^2 + x & x(x+2) = x^2 + 2x \\ x(x+3) = x^2 + 3x & x(x+4) = x^2 + 4x & (x+1)(x+1) = x^2 + 2x + 1 \\ (x+1)(x+2) = x^2 + 3x + 2 & (x+1)(x+3) = x^2 + 4x + 3 & (x+1)(x+4) = x^2 + 4 \\ (x+2)(x+2) = x^2 + 4x + 4 & (x+2)(x+3) = x^2 + 1 & (x+2)(x+4) = x^2 + x + 3 \\ (x+3)(x+3) = x^2 + x + 4 & (x+3)(x+4) = x^2 + 2x + 2 & (x+4)(x+4) = x^2 + 3x + 1 \end{array}$$

25. Make a list of all monic quadratics in $\mathbb{Z}_5[x]$.
[Answer:
$$\begin{array}{ccccc} x^2, & x^2 + 1, & x^2 + 2, & x^2 + 3, & x^2 + 4, \\ x^2 + x, & x^2 + x + 1, & x^2 + x + 2, & x^2 + x + 3, & x^2 + x + 4, \\ x^2 + 2x, & x^2 + 2x + 1, & x^2 + 2x + 2, & x^2 + 2x + 3, & x^2 + 2x + 4, \\ x^2 + 3x, & x^2 + 3x + 1, & x^2 + 3x + 2, & x^2 + 3x + 3, & x^2 + 3x + 4, \\ x^2 + 4x, & x^2 + 4x + 1, & x^2 + 4x + 2, & x^2 + 4x + 3, & x^2 + 4x + 4. \end{array}$$
We organize them by degree.]

26. Make a list of all irreducible monic quadratics in $\mathbb{Z}_5[x]$.
[Answer:

[4] Yes, we can still arrive at a monic as a product of two non-monic polynomials, but this is not an issue. If the coefficients, call them a and b multiply to one, then they are inverses of each other. We can then, multiply again by ba, to express the product as a product of monics.

$$\begin{array}{ccccc}
x^2, & x^2+1, & x^2+2, & x^2+3, & x^2+4, \\
x^2+x, & x^2+x+1, & x^2+x+2, & x^2+x+3, & x^2+x+4, \\
x^2+2x, & x^2+2x+1, & x^2+2x+2, & x^2+2x+3, & x^2+2x+4, \\
x^2+3x, & x^2+3x+1, & x^2+3x+2, & x^2+3x+3, & x^2+3x+4, \\
x^2+4x, & x^2+4x+1 & x^2+4x+2, & x^2+4x+3, & x^2+4x+4.
\end{array}$$

We simply remove the ones that came up as a product of two linear factors.]

27. We can also find the irreducible monic quadratics using the quadratic equation. In order to be irreducible, the discriminant of $b^2 - 4ac$ must not be a square. The squares in $\mathbb{Z}_5$ are $0, 1$ and 4. As $a = 1$, there are not many cases to check. Use this method to find all irreducible monic quadratics in $\mathbb{Z}_5[x]$.
[Answer: We need to avoid $b^2 + c = 0, b^2 + c = 1$ and $b^2 + c = 4$. This means c cannot equal $-b^2, 1 - b^2$ or $4 - b^2$.

b	$-b^2$	$1-b^2$	$4-b^2$	Remaining c
0	0	1	4	$2, 3$
1	4	0	3	$1, 2$
2	1	2	0	$3, 4$
3	1	2	0	$3, 4$
4	4	0	3	$1, 2$

We can read off the c values in the last column to get the irreducible quadratics for each b. This gives us:

$$\begin{array}{cc}
x^2+2 & x^2+3, \\
x^2+x+1 & x^2+x+2, \\
x^2+2x+3 & x^2+2x+4, \\
x^2+3x+3 & x^2+3x+4, \\
x^2+4x+1 & x^2+4x+2.]
\end{array}$$

5.4 Finding Irreducible Polynomials of Over $\mathbb{Z}_7$

1. How many monic quadratic irreducibles are there over $\mathbb{Z}_7$?
[Answer:

$$\frac{1}{2}\sum_{d|2}\mu(\frac{2}{d})7^d = \frac{1}{2}\left(\mu(2)7^1 + \mu(1)7^2\right) = \frac{1}{2}(-7+49) = \frac{42}{2} = 21.]$$

2. Find the product of these irreducibles.
[Answer:

$$\prod_{d|2}(x^{7^d} - x)^{\mu(\frac{2}{d})} = (x^{7^1} - x)^{\mu(\frac{2}{1})}(x^{7^2} - x)^{\mu(\frac{2}{2})} = (x^7 - x)^{-1}(x^{49} - x) =$$

$$\frac{x(x^{48}-1)}{x(x^6-1)} = \frac{x^{48}-1}{x^6-1} = x^{42} + x^{36} + x^{30} + x^{24} + x^{18} + x^{12} + x^6 + 1.]$$

3. Section off the terms into two groups to rewrite this as a product of smaller factors. You can do this twice.
[Answer:

$$x^{42} + x^{36} + x^{30} + x^{24} + x^{18} + x^{12} + x^6 + 1 = (x^{18} + x^{12} + x^6 + 1)x^{24} + (x^{18} + x^{12} + x^6 + 1) =$$

$$(x^{24} + 1)(x^{18} + x^{12} + x^6 + 1) = (x^{24} + 1)((x^6 + 1)x^{12} + x^6 + 1) = (x^{24} + 1)(x^{12} + 1)(x^6 + 1).]$$

4. Use the factoring rule for $x^{3a}+1$ to reduce each of these terms and then rewrite $(x^{24}+1)(x^{12}+1)(x^6+1)$.
[Answer:

$$(x^{24} + 1)(x^{12} + 1)(x^6 + 1) = (x^{16} - x^8 + 1)(x^8 + 1)(x^8 - x^4 + 1)(x^4 + 1)(x^4 - x^2 + 1)(x^2 + 1).]$$

5. Note that in $\mathbb{Z}_7$, $x^2 - x + 1 = x^2 + 6x + 8 = (x + 2)(x + 4)$. Use this on the factors with three terms to further simply our product of irreducibles.
[Answer: As $(x^{16} - x^8 + 1) = (x^8 + 2)(x^8 + 4), (x^8 - x^4 + 1) = (x^4 + 2)(x^4 + 4)$, and $(x^4 - x^2 + 1) = (x^2 + 2)(x^2 + 4)$, our product reduces to

$$(x^8 + 2)(x^8 + 4)(x^8 + 1)(x^4 + 2)(x^4 + 4)(x^4 + 1)(x^2 + 2)(x^2 + 4)(x^2 + 1).]$$

6. Next we attempt to factor $(x^4 + 1)$ (and thus also $(x^8 + 1)$) in $\mathbb{Z}_7$. Start by looking for a and b so that $(x^2 + ax + 1)(x^2 + bx + 1) = x^2 + 1$. What are the values that make this possible?
[Answer: We multiply out to get

$$x^2 + (a + b)x^3 + (ab + 2)x^2 + (a + b)x + 1.$$

As the coefficients of x, x^2, and x^3 must be zero, we know $a = -b$ and $ab + 2 = 0$. Thus $2 = b^2$ and b must be either 3 or 4. As $a = -b$ the possibilities are $a = 3, b = 4$ and $a = 4, b = 3$. These give us the factorization

$$(x^4 + 1) = (x^2 + 3x + 1)(x^2 + 4x + 1).$$

Plugging in x^2 also gives us

$$(x^8 + 1) = (x^4 + 3x^2 + 1)(x^4 + 4x^2 + 1).]$$

7. Attempt to factor $x^4 + 2$ by writing it as $(x^2 + ax + c)(x^2 + bx + d)$.
[Answer: The product gives us

$$x^4 + (a + b)x^3 + (ab + c + d)x^2 + (ad + bc)x + cd.$$

Thus $a = -b$, and we get the equations:

$$\begin{aligned} c + d - b^2 &= 0 \\ b(c - d) &= 0 \\ cd &= 2. \end{aligned}$$

If $b = 0$ then $c + d = 0$ so $d = -c$, and $-c^2 = 2$. Thus c^2 would have to be 5, which is not possible. As $b \neq 0$ we know $-d + c = 0$ and thus $d = c$. Now we know that $c^2 = 2$ so $c = 3$ or $c = 4$. If $c = 3$ we have $d = 3$, and $6 - b^2 = 0$ so $b^2 = 6$ which is not possible. If $c = 4$ we have $d = 4$, and $8 - b^2 = 1 - b^2 = 0$ so $b^2 = 1$. Thus $b = 1$ or $b = 6$. Our final possibilities are $a = 1, b = 6, c = 4, d = 4$ and $a = 6, b = 1, c = 4, d = 4$. We conclude that

$$(x^2 + x + 4)(x^2 - x + 4) = x^4 + 2.$$

This also tells us that

$$x^8 + 2 = (x^4 + x^2 + 4)(x^4 - x^2 + 4).$$

8. Attempt to factor $x^4 + 4$ by writing it as $(x^2 + ax + c)(x^2 + bx + d)$.
[Answer: The product gives us

$$x^4 + (a + b)x^3 + (ab + c + d)x^2 + (ad + bc)x + cd.$$

Thus $a = -b$, and we get the equations:

$$\begin{aligned} c + d - b^2 &= 0 \\ b(c - d) &= 0 \\ cd &= 4. \end{aligned}$$

If $b = 0$ then $c + d = 0$ so $d = -c$, and $-c^2 = 4$. Thus c^2 would have to be 3, which is not possible. As $b \neq 0$ we know $-d + c = 0$ and thus $d = c$. Now we know that $c^2 = 4$ so $c = 2$ or $c = 5$. If $c = 5$ we have $d = 5$, and $3 - b^2 = 0$ so $b^2 = 3$ which is not possible. If $c = 2$ we have $d = 2$, and $4 - b^2 = 0$ so $b^2 = 4$[5]. Thus $b = 2$ or $b = 5$. Our final possibilities are $a = 2, b = 5, c = 2, d = 2$ and $a = 5, b = 2, c = 2, d = 2$. We conclude that

$$(x^2 + 2x + 2)(x^2 - 2x + 2) = x^4 + 2.$$

This also tells us that

$$x^8 + 2 = (x^4 + 2x^2 + 2)(x^4 - 2x^2 + 2).$$

9. Use the last factorizations to rewrite the product of irreducible monic quadratics in $\mathbb{Z}_7$.
[Answer:

$$(x^4 + 2x^2 + 2)(x^4 + 5x^2 + 2)(x^4 + x^2 + 4)(x^4 + 6x^2 + 4)(x^4 + 3x^2 + 1)(x^4 + 4x^2 + 1)\cdot$$

$$(x^2 + 2x + 2)(x^2 + 5x + 2)(x^2 + x + 4)(x^2 + 6x + 4)(x^2 + 3x + 1)(x^2 + 4x + 1)(x^2 + 2)(x^2 + 4)(x^2 + 1).]$$

10. Factor the remaining quartics in our product in $\mathbb{Z}_7[x]$.
[Answer: For $x^4 + ex^2 + f$ we must solve

$$x^4 + ex^2 + f = x^4 + (a + b)x^3 + (ab + c + d)x^2 + (ad + bc)x + cd.$$

We have $a + b = 0$ so $a = -b$, and we get

[5] Notice that unlike in the similar computations, when we arrived at $4 - b^2 = 0$, we did not have to reduce the 4 modulo seven. This actually works for general polynomials over $\mathbb{R}[x]$ so long as we decide not to replace -2 with 5 in the end.

$$\begin{aligned} c + d - b^2 &= e \\ b(c - d) &= 0 \\ cd &= f. \end{aligned}$$

We need to solve the cases $e = \pm 2, f = 2$ and $e = \pm 1, f = 4$ and $e = \pm 3, f = 1$. If $b = 0$ then we get that c must satisfy $c(e - c) = f$ so $c^2 - ec + f$. In our six cases for e and f these lead to the equations

$$\begin{array}{cc} c^2 - 2c + 2 = 0 & c^2 + 2c + 2 = 0 \\ c^2 - c + 4 = 0 & c^2 + c + 4 = 0 \\ c^2 + 3c + 1 = 0 & c^2 - 3c + 1 = 0. \end{array}$$

These are exactly the irreducible quadratics we can already see in our product. Hence none of these are solvable for c and therefore b can not be zero. This tells us that $c - d = 0$ so $c = d$ and thus $c^2 = f$. For $f = 1$ we know $c = d = 1$ or $c = d = 6$. For $f = 2$ we know $c = d = 3$ or $c = d = 4$. For $f = 4$ we know $c = d = 2$ or $c = d = 5$. We can then use $b^2 = 2c - e$ in each of these cases.

f	e	c,d	$2c - e$	f	e	c,d	$2c - e$	f	e	c,d	$2c - e$
1	3	1	6	2	2	3	4	4	1	2	3
1	3	6	2	2	2	4	6	4	1	5	2
1	4	1	5	2	5	3	1	4	6	2	5
1	4	6	1	2	5	4	3	4	6	5	4

Half of these are not possible as $2c - e$ can not be a square. This leaves the following cases.

f	e	c,d	$2c - e$	a,b	f	e	c,d	$2c - e$	a,b	f	e	c,d	$2c - e$	a,b
1	3	6	2	3, 4	2	2	3	4	2, 5	4	1	5	2	3, 4
1	4	6	1	1, 6	2	5	3	1	1, 6	4	6	5	4	2, 5.

We can write out each of these to arrive at

$$\begin{gathered} (x^4 + 3x^2 + 1) = (x^2 + 3x + 6)(x^2 + 4x + 6) \\ (x^4 + 4x^2 + 1) = (x^2 + x + 6)(x^2 + 6x + 6) \\ (x^4 + 2x^2 + 2) = (x^2 + 2x + 3)(x^2 + 5x + 3) \\ (x^4 + 5x^2 + 2) = (x^2 + x + 3)(x^2 + 6x + 3) \\ (x^4 + x^2 + 4) = (x^2 + 3x + 5)(x^2 + 4x + 5) \\ (x^4 + 6x^2 + 4) = (x^2 + 2x + 5)(x^2 + 5x + 5).] \end{gathered}$$

11. Write out the fully factored version of $\frac{x^{48}-1}{x^6-1}$ thus allowing us to read off the irreducibles from factorization tricks alone.
[Answer:

$$(x^2 + 3x + 6)(x^2 + 4x + 6)(x^2 + x + 6)(x^2 + 6x + 6)(x^2 + 2x + 3)(x^2 + 5x + 3)(x^2 + x + 3)\cdot$$

$$(x^2 + 6x + 3)(x^2 + 3x + 5)(x^2 + 4x + 5)(x^2 + 2x + 5)(x^2 + 5x + 5)(x^2 + 2x + 2)(x^2 + 5x + 2)\cdot$$

$$(x^2 + x + 4)(x^2 + 6x + 4)(x^2 + 3x + 1)(x^2 + 4x + 1)(x^2 + 2)(x^2 + 4)(x^2 + 1).]$$

12. Write $p(x) = x^2 + bx + c$, and use the fact that $p(x) \neq 0$ for any $x \in \mathbb{Z}_7$ if $p(x)$ is irreducible, to find the irreducible quadratics in $\mathbb{Z}_7[x]$.
[Answer: Plugging in the numbers 0 through 6 tells us that the following seven equations must not equal zero for any b and c in an irreducible:

x	Equation	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$b=6$
0	$c \neq 0$	$c \neq 0$	$c \neq 0$	$c \neq 0$	$c \neq 0$	$c \neq 0$	$c \neq 0$	$c \neq 0$
1	$1+b+c \neq 0$	$c \neq 6$	$c \neq 5$	$c \neq 4$	$c \neq 3$	$c \neq 2$	$c \neq 1$	$c \neq 0$
2	$4+2b+c \neq 0$	$c \neq 3$	$c \neq 1$	$c \neq 6$	$c \neq 4$	$c \neq 2$	$c \neq 0$	$c \neq 5$
3	$2+3b+c \neq 0$	$c \neq 5$	$c \neq 2$	$c \neq 6$	$c \neq 3$	$c \neq 0$	$c \neq 4$	$c \neq 1$
4	$2+4b+c \neq 0$	$c \neq 5$	$c \neq 1$	$c \neq 4$	$c \neq 0$	$c \neq 3$	$c \neq 6$	$c \neq 2$
5	$4+5b+c \neq 0$	$c \neq 3$	$c \neq 5$	$c \neq 0$	$c \neq 2$	$c \neq 4$	$c \neq 6$	$c \neq 1$
6	$1+6b+c \neq 0$	$c \neq 6$	$c \neq 0$	$c \neq 1$	$c \neq 2$	$c \neq 3$	$c \neq 4$	$c \neq 5$

We can now immediately read off which values of c produce irreducibles for the different values of b.

$$\begin{array}{ccc}
x^2+1, & x^2+2, & x^2+4, \\
x^2+x+3, & x^2+x+4, & x^2+x+6, \\
x^2+2x+2, & x^2+2x+3, & x^2+2x+5, \\
x^2+3x+1, & x^2+3x+5, & x^2+3x+6, \\
x^2+4x+1, & x^2+4x+5, & x^2+4x+6, \\
x^2+5x+2, & x^2+5x+3, & x^2+5x+5, \\
x^2+6x+3, & x^2+6x+4, & x^2+6x+6.]
\end{array}$$

13. Find the irreducible quadratics by taking all possible combinations of two linear factors, and removing them from the list of all possibilities.
[Answer: We first list every possible product of two linear factors.

$$\begin{array}{ccc}
x \cdot x = x^2 & (x+1)(x+1) = x^2+2x+1 & (x+2)(x+2) = x^2+4x+4 \\
(x+3)(x+3) = x^2+6x+2 & (x+4)(x+4) = x^2+x+2 & (x+5)(x+5) = x^2+3x+4 \\
(x+6)(x+6) = x^2+5x+1 & x(x+1) = x^2+x & x(x+2) = x^2+2x \\
x(x+3) = x^2+3x & x(x+4) = x^2+4x & x(x+5) = x^2+5x \\
x(x+6) = x^2+6x & (x+1)(x+2) = x^2+3x+2 & (x+1)(x+3) = x^2+4x+3 \\
(x+1)(x+4) = x^2+5x+4 & (x+1)(x+5) = x^2+6x+5 & (x+1)(x+6) = x^2+6 \\
(x+2)(x+3) = x^2+5x+6 & (x+2)(x+4) = x^2+6x+1 & (x+2)(x+5) = x^2+3 \\
(x+2)(x+6) = x^2+x+5 & (x+3)(x+4) = x^2+5 & (x+3)(x+5) = x^2+x+1 \\
(x+3)(x+6) = x^2+2x+4 & (x+4)(x+5) = x^2+2x+6 & (x+4)(x+6) = x^2+3x+3 \\
 & (x+5)(x+6) = x^2+4x+2 &
\end{array}$$

We then consider all possible quadratics.

$$\begin{array}{ccccccc}
x^2, & x^2+1, & x^2+2, & x^2+3, & x^2+4, & x^2+5, & x^2+6, \\
x^2+x, & x^2+x+1, & x^2+x+2, & x^2+x+3, & x^2+x+4, & x^2+x+5, & x^2+x+6, \\
x^2+2x, & x^2+2x+1, & x^2+2x+2, & x^2+2x+3, & x^2+2x+4, & x^2+2x+5, & x^2+2x+6, \\
x^2+3x, & x^2+3x+1, & x^2+3x+2, & x^2+3x+3, & x^2+3x+4, & x^2+3x+5, & x^2+3x+6, \\
x^2+4x, & x^2+4x+1, & x^2+4x+2, & x^2+4x+3, & x^2+4x+4, & x^2+4x+5, & x^2+4x+6, \\
x^2+5x, & x^2+5x+1, & x^2+5x+2, & x^2+5x+3, & x^2+5x+4, & x^2+5x+5, & x^2+5x+6, \\
x^2+6x, & x^2+6x+1, & x^2+6x+2, & x^2+6x+3, & x^2+6x+4, & x^2+6x+5, & x^2+6x+6.
\end{array}$$

Finally, we remove the ones we know to be irreducible.

$$\begin{array}{ccccccc}
x^2, & \boldsymbol{x^2+1}, & \boldsymbol{x^2+2}, & x^2+3, & \boldsymbol{x^2+4}, & x^2+5, & x^2+6, \\
x^2+x, & x^2+x+1, & x^2+x+2, & \boldsymbol{x^2+x+3}, & \boldsymbol{x^2+x+4}, & x^2+x+5, & \boldsymbol{x^2+x+6}, \\
x^2+2x, & x^2+2x+1, & \boldsymbol{x^2+2x+2}, & \boldsymbol{x^2+2x+3}, & x^2+2x+4, & \boldsymbol{x^2+2x+5}, & x^2+2x+6, \\
x^2+3x, & \boldsymbol{x^2+3x+1}, & x^2+3x+2, & x^2+3x+3, & x^2+3x+4, & \boldsymbol{x^2+3x+5}, & \boldsymbol{x^2+3x+6}, \\
x^2+4x, & \boldsymbol{x^2+4x+1}, & x^2+4x+2, & x^2+4x+3, & x^2+4x+4, & \boldsymbol{x^2+4x+5}, & \boldsymbol{x^2+4x+6}, \\
x^2+5x, & x^2+5x+1, & \boldsymbol{x^2+5x+2}, & \boldsymbol{x^2+5x+3}, & x^2+5x+4, & \boldsymbol{x^2+5x+5}, & x^2+5x+6, \\
x^2+6x, & x^2+6x+1, & x^2+6x+2, & \boldsymbol{x^2+6x+3}, & \boldsymbol{x^2+6x+4}, & x^2+6x+5, & \boldsymbol{x^2+6x+6}.]
\end{array}$$

14. Us the fact that the discriminant of b^2-4ac must not be a square in an irreducible quadratic to find all irreducible monic quadratics in $\mathbb{Z}_7[x]$.
[Answer: The squares in $\mathbb{Z}_7$ are 0, 1, 2, and 4. We set $a=1$ to examine monic polynomials only. We must avoid $b^2+3c=0, b^2+3c=1, b^2+3c=2$ and $b^2+3c=4$. This means $3c$ cannot equal $-b^2, 1-b^2, 2-b^2$ or $4-b^2$ and multiplying by five tells us c cannot equal $2b^2, 5+2b^2, 3+2b^2$ or $6+2b^2$.

b	$2b^2$	$6+2b^2$	$5+2b^2$	$3+2b^2$	Remaining c
0	0	6	5	3	$1,2,4$
1	2	1	0	5	$3,4,6$
2	1	0	6	4	$2,3,5$
3	4	3	2	0	$1,5,6$
4	4	3	2	0	$1,5,6$
5	1	0	6	4	$2,3,5$
6	2	1	0	5	$3,4,6$

We can read off the c values in the last column to get the irreducible quadratics for each b. This gives us:

$$\begin{array}{ccc}
x^2+1 & x^2+2 & x^2+4 \\
x^2+x+3 & x^2+x+4 & x^2+x+6 \\
x^2+2x+2 & x^2+2x+3 & x^2+2x+5 \\
x^2+3x+1 & x^2+3x+5 & x^2+3x+6 \\
x^2+4x+1 & x^2+4x+5 & x^2+4x+6 \\
x^2+5x+2 & x^2+5x+3 & x^2+5x+5 \\
x^2+6x+3 & x^2+6x+4 & x^2+6x+6.]
\end{array}$$

5.5 Finding Irreducible Polynomials of Over $\mathbb{Z}_{11}$

1. How many monic quadratic irreducibles are there over $\mathbb{Z}_{11}$?
[Answer:
$$\frac{1}{2}\sum_{d|2}\mu(\frac{2}{d})11^d = \frac{1}{2}\left(\mu(2)11^1+\mu(1)11^2\right) = \frac{1}{2}\left(-11+121\right) = \frac{110}{2} = 55.]$$

2. Find the product of these irreducibles.
[Answer:
$$\prod_{d|2}(x^{11^d}-x)^{\mu(\frac{2}{d})} = (x^{11^1}-x)^{\mu(\frac{2}{1})}(x^{11^2}-x)^{\mu(\frac{2}{2})} = (x^{11}-x)^{-1}(x^{121}-x) =$$

$$\frac{x(x^{120}-1)}{x(x^{10}-1)} = \frac{x^{120}-1}{x^{10}-1} = x^{110}+x^{100}+x^{90}+x^{80}+x^{70}+x^{60}+x^{50}+x^{40}+x^{30}+x^{20}+x^{10}+1.]$$

3. Use grouping techniques to further simplify this polynomial.
[Answer:
$$\begin{gathered} x^{110}+x^{100}+x^{90}+x^{80}+x^{70}+x^{60}+x^{50}+x^{40}+x^{30}+x^{20}+x^{10}+1 = \\ x^{100}(x^{10}+1)+x^{80}(x^{10}+1)+x^{60}(x^{10}+1)+x^{40}(x^{10}+1)+x^{20}(x^{10}+1)+(x^{10}+1) = \\ (x^{100}+x^{80}+x^{60}+x^{40}+x^{20}+1)(x^{10}+1) = \\ (x^{80}(x^{20}+1)+x^{40}(x^{20}+1)+1(x^{20}+1))(x^{10}+1) = \\ (x^{80}+x^{40}+1)(x^{20}+1)(x^{10}+1).] \end{gathered}$$

4. Use the formula for $x^{4a}+x^{2a}+1$ to further simplify $x^{80}+x^{40}+1$.
[Answer:
$$\begin{gathered} x^{80}+x^{40}+1 = x^{80}+x^{60}+x^{40}-x^{60}-x^{40}-x^{20}+x^{40}+x^{20}+1 = \\ x^{40}(x^{40}+x^{20}+1)-x^{20}(x^{40}+x^{20}+1)+(x^{40}+x^{20}+1) = (x^{40}-x^{20}+1)(x^{40}+x^{20}+1).] \end{gathered}$$

5. Use the formula for $x^{4a}+x^{2a}+1$ to further simplify $x^{40}+x^{20}+1$.
[Answer:
$$\begin{gathered} x^{40}+x^{20}+1 = x^{40}+x^{30}+x^{20}-x^{30}-x^{20}-x^{10}+x^{20}+x^{10}+1 = \\ x^{20}(x^{20}+x^{10}+1)-x^{10}(x^{20}+x^{10}+1)+(x^{20}+x^{10}+1) = (x^{20}-x^{10}+1)(x^{20}+x^{10}+1).] \end{gathered}$$

6. Use the formula yet again to further simplify $x^{20}+x^{10}+1$.
[Answer:
$$\begin{gathered} x^{20}+x^{10}+1 = x^{20}+x^{15}+x^{10}-x^{15}-x^{10}-x^{5}+x^{10}+x^{5}+1 = \\ x^{10}(x^{10}+x^{5}+1)-x^{5}(x^{10}+x^{5}+1)+(x^{10}+x^{5}+1) = (x^{10}-x^{5}+1)(x^{10}+x^{5}+1).] \end{gathered}$$

7. Put these factorizations together to rewrite the product of irreducibles over $\mathbb{Z}_{11}$. Note that this is as far as we can factor in $\mathbb{R}$ so we must use the fact that out field is finite if we wish to go further.
[Answer: Our factored form is
$$(x^{20}+1)(x^{10}+1)(x^{40}-x^{20}+1)(x^{20}-x^{10}+1)(x^{10}-x^{5}+1)(x^{10}+x^{5}+1).]$$

8. Make a table of the values of b^2-4c for all b and c in $\mathbb{Z}_{11}$.
[Answer:

		c										
		0	1	2	3	4	5	6	7	8	9	10
	0	0	7	3	10	6	2	9	5	1	8	4
	1	1	8	4	0	7	3	10	6	2	9	5
	2	4	0	7	3	10	6	2	9	5	1	8
	3	9	5	1	8	4	0	7	3	10	6	2
	4	5	1	8	4	0	7	3	10	6	2	9
b	5	3	10	6	2	9	5	1	8	4	0	7
	6	3	10	6	2	9	5	1	8	4	0	7
	7	5	1	8	4	0	7	3	10	6	2	9
	8	9	5	1	8	4	0	7	3	10	6	2
	9	4	0	7	3	10	6	2	9	5	1	8
	10	1	8	4	0	7	3	10	6	2	9	5

.]

9. Use this table to find all values of b and c so that $b^2 - 4c$ is not a square in $\mathbb{Z}_{11}$.
[Answer: The squares are $\{0, 1, 3, 4, 5, 9\}$, so we get a list of (b, c) pairs equal to :

$$(0,1),(0,3),(0,4),(0,5),(0,9),$$

$$(1,1),(1,4),(1,6),(1,7),(1,8),$$
$$(2,2),(2,4),(2,5),(2,6),(2,10),$$
$$(3,3),(3,6),(3,8),(3,9),(3,10),$$
$$(4,2),(4,5),(4,7),(4,8),(4,9),$$
$$(5,1),(5,2),(5,3),(5,7),(5,10),$$
$$(6,1),(6,2),(6,3),(6,7),(6,10),$$
$$(7,2),(7,5),(7,7),(7,8),(7,9),$$
$$(8,3),(8,6),(8,8),(8,9),(8,10),$$
$$(9,2),(9,4),(9,5),(9,6),(9,10),$$
$$(10,1),(10,4),(10,6),(10,7),(10,8).]$$

10. Use this list of pairs to find all irreducible quadratics over $\mathbb{Z}_{11}$.
[Answer: The squares are $\{0, 1, 3, 4, 5, 9\}$, so we get a list of (b, c) pairs equal to :

$$(x^2+1),(x^2+3),(x^2+4),(x^2+5),(x^2+9),$$

$$(x^2+1x+1),(x^2+1x+4),(x^2+1x+6),(x^2+1x+7),(x^2+1x+8),$$
$$(x^2+2x+2),(x^2+2x+4),(x^2+2x+5),(x^2+2x+6),(x^2+2x+10),$$
$$(x^2+3x+3),(x^2+3x+6),(x^2+3x+8),(x^2+3x+9),(x^2+3x+10),$$
$$(x^2+4x+2),(x^2+4x+5),(x^2+4x+7),(x^2+4x+8),(x^2+4x+9),$$
$$(x^2+5x+1),(x^2+5x+2),(x^2+5x+3),(x^2+5x+7),(x^2+5x+10),$$
$$(x^2+6x+1),(x^2+6x+2),(x^2+6x+3),(x^2+6x+7),(x^2+6x+10),$$
$$(x^2+7x+2),(x^2+7x+5),(x^2+7x+7),(x^2+7x+,8),(x^2+7x+9),$$
$$(x^2+8x+3),(x^2+8x+6),(x^2+8x+8),(x^2+8x+9),(x^2+8x+10),$$
$$(x^2+9x+2),(x^2+9x+4),(x^2+9x+5),(x^2+9x+6),(x^2+9x+10),$$
$$(x^2+10x+1),(x^2+10x+4),(x^2+10x+6),(x^2+10x+7),(x^2+10x+8).]$$

Chapter 6

Construction of Finite Fields

6.1 Finite Fields of Order 2^n.

1. Use the irreducible quadratic $x^2 + x + 1$ in $\mathbb{Z}_2[x]$ to construct the finite field of order four. Write out the addition and multiplication tables for this field.

   [Answer: We introduce an element a that satisfies $a^2 + a + 1 = 0$. Here $a^2 = a + 1$.

$+$	0	1	a	$a+1$
0	0	1	a	$a+1$
1	1	0	$a+1$	a
a	a	$a+1$	0	1
$a+1$	$a+1$	a	1	0

$\cdot$	0	1	a	$a+1$
0	0	0	0	0
1	0	1	a	$a+1$
a	0	a	$a+1$	1
$a+1$	0	$a+1$	1	a

.]

2. Use the irreducible cubic $x^3 + x^2 + 1$ in $\mathbb{Z}_2[x]$ to construct the finite field of order eight.
   [Answer: We introduce an element a that satisfies $a^3 + a^2 + 1 = 0$. Here $a^3 = a^2 + 1$.

$+$	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
0	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
1	1	0	$a+1$	a	a^2+1	a^2	a^2+a+1	a^2+a
a	a	$a+1$	0	1	a^2+a	a^2+a+1	a^2	a^2+1
$a+1$	$a+1$	a	1	0	a^2+a+1	a^2+a	a^2+1	a^2
a^2	a^2	a^2+1	a^2+a	a^2+a+1	0	1	a	$a+1$
a^2+1	a^2+1	a^2	a^2+a+1	a^2+a	1	0	$a+1$	a
a^2+a	a^2+a	a^2+a+1	a^2+1	a^2	a	$a+1$	0	1
a^2+a+1	a^2+a+1	a^2+a	a^2	a^2+1	$a+1$	1	1	0

$\cdot$	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
0	0	0	0	0	0	0	0	0
1	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
a	0	a	a^2	a^2+a	a^2+1	a^2+a+1	1	$a+1$
$a+1$	0	$a+1$	a^2+a	a^2+1	1	a	a^2+a+1	a^2
a^2	0	a^2	a^2+1	1	a^2+a+1	$a+1$	a	a^2+a
a^2+1	0	a^2+1	a^2+a+1	a	$a+1$	a^2+a	a^2	1
a^2+a	0	a^2+a	1	a^2+a+1	a	a^2	$a+1$	a^2+1
a^2+a+1	0	a^2+a+1	$a+1$	a^2	a^2+a	1	a^2+1	a

.]

3. Use the irreducible cubic x^3+x+1 in $\mathbb{Z}_2[x]$ to construct the finite field of order eight.
[Answer: Here we get the same table for addition as before. We introduce an element satisfying a^3+a+1 and thus have $a^3 = a+1$. Four our multiplication table we get

$\cdot$	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
0	0	0	0	0	0	0	0	0
1	0	1	a	$a+1$	a^2	a^2+1	a^2+a	a^2+a+1
a	0	a	a^2	a^2+a	$a+1$	1	a^2+a+1	a^2+1
$a+1$	0	$a+1$	a^2+a	a^2+1	a^2+a+1	a^2	1	a
a^2	0	a^2	$a+1$	a^2+a+1	a^2+a	a	a^2+1	1
a^2+1	0	a^2+1	1	a^2	a	a^2+a+1	$a+1$	a^2+a
a^2+a	0	a^2+a	a^2+a+1	1	a^2+1	$a+1$	a	a^2
a^2+a+1	0	a^2+a+1	a^2+1	a	1	a^2+a	a^2	$a+1$

.]

4. For the finite field $GF(16)$ constructed from the irreducible x^4+x+1 in $\mathbb{Z}_2[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2
0	0	a^3	a^3+a^2
1	1	a^3+1	a^3+a^2+1
a	a^2	a^3+a	a^3
$a+1$	a^2+1	a^3+a+1	a^3+1
a^2	$a+1$	a^3+a^2	a^3+a^2+a+1
a^2+1	a	a^3+a^2+1	a^3+a^2+a
a^2+a	a^2+a+1	a^3+a^2+a	a^3+a+1
a^2+a+1	a^2+a	a^3+a^2+a+1	a^3+a

5. For the finite field $GF(16)$ constructed from the irreducible x^4+x^3+1 in $\mathbb{Z}_2[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2
0	0	a^3	a^3+a^2+a+1
1	1	a^3+1	a^3+a^2+a
a	a^2	a^3+a	a^3+a+1
$a+1$	a^2+1	a^3+a+1	a^3+a
a^2	a^3+1	a^3+a^2	a^2+a
a^2+1	a^3	a^3+a^2+1	a^2+a+1
a^2+a	a^3+a^2+1	a^3+a^2+a	a
a^2+a+1	a^3+a^2	a^3+a^2+a+1	$a+1$

.]

6. For the finite field $GF(16)$ constructed from the irreducible $x^4+x^3+x^2+x+1$ in $\mathbb{Z}_2[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2
0	0	a^3	a
1	1	a^3+1	$a+1$
a	a^2	a^3+a	a^2+a
$a+1$	a^2+1	a^3+a+1	a^2+a+1
a^2	a^3+a^2+a+1	a^3+a^2	a^3+a^2+1
a^2+1	a^3+a^2+a	a^3+a^2+1	a^3+a^2
a^2+a	a^3+a+1	a^3+a^2+a	a^3+1
a^2+a+1	a^3+a	a^3+a^2+a+1	a^3

.]

6.2 Finite Fields of Order 3^n.

1. Make a character table for $GF(3)$.
[Answer:

	0	1	2
0	0	-1	1
1	1	0	-1
2	-1	1	0

.]

2. Use this character table to draw the Paley three graph.
[Answer: See figure 6.1.]

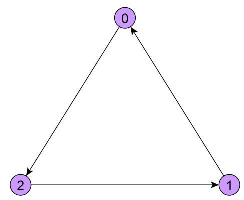


Figure 6.1: Paley Graph on Three Vertices

3. Use the irreducible cubic $x^2 + 1$ in $\mathbb{Z}_3[x]$ to construct the finite field of order nine.
[Answer: We introduce an element a that satisfies $a^2 + 1 = 0$. Here $a^2 = -1 = 2$.

$+$	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
1	1	2	0	$a+1$	$a+2$	a	$2a+1$	$2a+2$	$2a$
2	2	0	1	$a+2$	a	$a+1$	$2a+2$	$2a$	$2a+1$
a	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$	0	1	2
$a+1$	$a+1$	$a+2$	a	$2a+1$	$2a+2$	$2a$	1	2	0
$a+2$	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$	2	0	1
$2a$	$2a$	$2a+1$	$2a+2$	0	1	2	a	$a+1$	$a+2$
$2a+1$	$2a+1$	$2a+2$	$2a$	1	2	0	$a+1$	$a+2$	a
$2a+2$	$2a$	$2a+1$	$2a+2$	2	0	1	a	$a+1$	$a+2$

$\cdot$	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
2	0	2	1	$2a$	$2a+2$	$2a+1$	a	$a+2$	$a+1$
a	0	a	$2a$	2	$a+2$	$2a+2$	1	$a+1$	$2a+1$
$a+1$	0	$a+1$	$2a+2$	$a+2$	$2a$	1	$2a+1$	2	a
$a+2$	0	$a+2$	$2a+1$	$2a+2$	1	a	$a+1$	$2a$	2
$2a$	0	$2a$	a	1	$2a+1$	$a+1$	2	$2a+2$	$a+2$
$2a+1$	0	$2a+1$	$a+2$	$a+1$	2	$2a$	$2a+2$	a	1
$2a+2$	0	$2a+2$	$a+1$	$2a+1$	a	2	$a+2$	1	$2a$

.]

4. Which elements here are squares?
   [Answer: $0, 1, 2, a$, and $2a$.]

5. Make a subtraction table for the $x^2 + 1$ construction the finite field of order nine.
   [Answer: We introduce an element a that satisfies $a^2 + 1 = 0$. Here $a^2 = -1 = 2$.

$-$	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	2	1	$2a$	$2a+2$	$2a+1$	a	$a+2$	$a+1$
1	1	0	2	$2a+1$	$2a$	$2a+2$	$a+1$	a	$a+2$
2	2	1	0	$2a+2$	$2a+1$	$2a$	$a+2$	$a+1$	a
a	a	$a+2$	$a+1$	0	2	1	$2a$	$2a+2$	$2a+1$
$a+1$	$a+1$	a	$a+2$	1	0	2	$2a+1$	$2a$	$2a+2$
$a+2$	$a+2$	$a+1$	a	2	1	0	$2a+2$	$2a+1$	$2a$
$2a$	$2a$	$2a+2$	$2a+1$	a	$a+2$	$a+1$	0	2	1
$2a+1$	$2a+1$	$2a$	$2a+2$	$a+1$	a	$a+2$	1	0	2
$2a+2$	$2a+2$	$2a+1$	$2a$	$a+2$	$a+1$	a	2	1	0

.]

6. Find the character table for this construction of $GF(9)$. Recall the entry corresponding to the row for x and column for y equals one if $x - y$ is a non-zero square, minus one if it is not a square, and equals zero if $x - y$ is zero.
   [Answer:

χ	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	1	1	1	-1	-1	1	-1	-1
1	1	0	1	-1	1	-1	-1	1	-1
2	1	1	0	-1	-1	1	-1	-1	1
a	1	-1	-1	0	1	1	1	-1	-1
$a+1$	-1	1	-1	1	0	1	-1	1	-1
$a+2$	-1	-1	1	1	1	0	-1	-1	1
$2a$	1	-1	-1	1	-1	-1	0	1	1
$2a+1$	-1	1	-1	-1	1	-1	1	0	1
$2a+2$	-1	-1	1	-1	-1	1	1	1	0

.]

7. Use this character table to draw the Paley nine graph.
   [Answer: See figure 6.2.]

8. Use the irreducible cubic $x^2 + x + 2$ in $\mathbb{Z}_3[x]$ to construct the finite field of order nine.
   [Answer: We introduce an element a that satisfies $a^2 + a + 2 = 0$. Here $a^2 = -a - 2 = 2a + 1$. Our

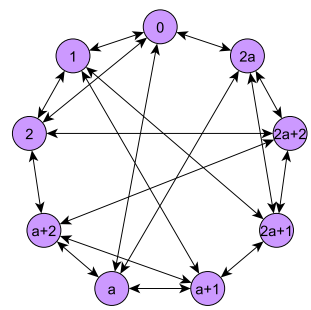


Figure 6.2: Paley Graph on Nine Vertices

addition table is the same, but we arrive at a new multiplication as follows:

$\cdot$	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
2	0	2	1	$2a$	$2a+2$	$2a+1$	a	$a+2$	$a+1$
a	0	a	$2a$	$2a+1$	1	$a+1$	$a+2$	$2a+2$	2
$a+1$	0	$a+1$	$2a+2$	1	$a+2$	$2a$	2	a	$2a+1$
$a+2$	0	$a+2$	$2a+1$	$a+1$	$2a$	2	$2a+2$	1	a
$2a$	0	$2a$	a	$a+2$	2	$2a+2$	$2a+1$	$a+1$	1
$2a+1$	0	$2a+1$	$a+2$	$2a+2$	a	1	$a+1$	2	$2a$
$2a+2$	0	$2a+2$	$a+1$	2	$2a+1$	a	1	$2a$	$a+2$

.]

9. Which elements here are squares?
[Answer: $0, 1, 2, a+2$, and $2a+1$.]

10. Find the character table for this construction of $GF(9)$.
[Answer:

χ	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	1	1	-1	-1	1	-1	1	-1
1	1	0	1	1	-1	-1	-1	-1	1
2	1	1	0	-1	1	-1	1	-1	-1
a	-1	1	-1	0	1	1	-1	-1	1
$a+1$	-1	-1	1	1	0	1	1	-1	-1
$a+2$	1	-1	-1	1	1	0	-1	1	-1
$2a$	-1	-1	1	-1	1	-1	0	1	1
$2a+1$	1	-1	-1	-1	-1	1	1	0	1
$2a+2$	-1	1	-1	1	-1	-1	1	1	0

.]

11. Use this character table to draw the Paley nine graph[1].
[Answer: See figure 6.3.]

[1] This graph will be isomorphic to the other construction, though the vertices will be labeled differently.

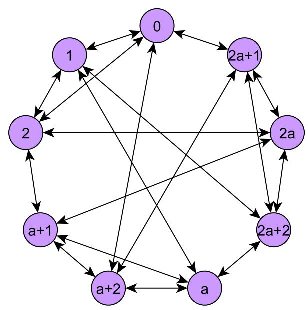


Figure 6.3: Paley Graph on Nine Vertices

12. Use the irreducible cubic $x^2 - x + 2$ in $\mathbb{Z}_3[x]$ to construct the finite field of order nine.
[Answer: We introduce an element a that satisfies $a^2 - a + 2 = 0$. Here $a^2 = a - 2 = a + 1$. Our addition table is the same, but we arrive at a new multiplication as follows:

$\cdot$	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
2	0	2	1	$2a$	$2a+2$	$2a+1$	a	$a+2$	$a+1$
a	0	a	$2a$	$a+1$	$2a+1$	1	$2a+2$	2	$a+2$
$a+1$	0	$a+1$	$2a+2$	$2a+1$	2	a	$a+2$	$2a$	1
$a+2$	0	$a+2$	$2a+1$	1	a	$2a+2$	2	$a+1$	$2a$
$2a$	0	$2a$	a	$2a+2$	$a+2$	2	$a+1$	1	$2a+1$
$2a+1$	0	$2a+1$	$a+2$	2	$2a$	$a+1$	1	$2a+2$	a
$2a+2$	0	$2a+2$	$a+1$	$a+2$	1	$2a$	$2a+1$	a	2

.]

13. Which elements here are squares?
[Answer: $0, 1, 2, a+1$, and $2a+2$.]

14. Find the character table for this construction of $GF(9)$.
[Answer:

χ	0	1	2	a	$a+1$	$a+2$	$2a$	$2a+1$	$2a+2$
0	0	1	1	-1	1	-1	-1	-1	1
1	1	0	1	-1	-1	1	1	-1	-1
2	1	1	0	1	-1	-1	-1	1	-1
a	-1	-1	1	0	1	1	-1	1	-1
$a+1$	1	-1	-1	1	0	1	-1	-1	1
$a+2$	-1	1	-1	1	1	0	1	-1	-1
$2a$	-1	1	-1	-1	-1	1	0	1	1
$2a+1$	-1	-1	1	1	-1	-1	1	0	1
$2a+2$	1	-1	-1	-1	1	-1	1	1	0

.]

15. Use this character table to draw the Paley nine graph[2].

[2] Again, only the labels should be different from our other Paley nine graphs.

[Answer: See figure 6.4.]

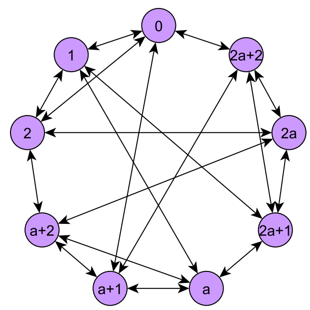


Figure 6.4: Paley Graph on Nine Vertices

16. For the finite field constructed from the irreducible x^3+2x+1 in $\mathbb{Z}_3[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2
0	0	a^2	a^2+2a	$2a^2$	a^2+2a
1	1	a^2+1	$2a+1$	$2a^2+1$	$2a^2+2a+1$
2	1	a^2+2	$2a^2+2a+1$	$2a^2+2$	$2a+1$
a	a^2	a^2+a	$2a^2+a+1$	$2a^2+a$	$2a^2+2$
$a+1$	a^2+2a+1	a^2+a+1	a^2+2	$2a^2+a+1$	$2a$
$a+2$	a^2+a+1	a^2+a+2	$2a+2$	$2a^2+a+2$	a^2+a
$2a$	a^2	a^2+2a	$2a^2+2$	$2a^2+2a$	$2a^2+a+1$
$2a+1$	a^2+a+1	a^2+2a+1	a^2+a	$2a^2+2a+1$	$2a+2$
$2a+2$	a^2+2a+1	a^2+2a+2	$2a$	$2a^2+2a+2$	a^2+2

.]

17. Use your table to list the squares in this construction of $GF(27)$.
[Answer: $0, 1, 2x, 2x+1, 2x+2, x^2, x^2+2, x^2+x, x^2+x+1, x^2+2x, x^2+2x+1, 2x^2+2, 2x^2+x+1, 2x^2+2x+1$.]

18. For the finite field constructed from the irreducible x^3+2x+2 in $\mathbb{Z}_3[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2
0	0	a^2	a^2+a	$2a^2$	a^2+a
1	1	a^2+1	$a+1$	$2a^2+1$	$2a^2+a+1$
2	1	a^2+2	$2a^2+a+1$	$2a^2+2$	$a+1$
a	a^2	a^2+a	$2a^2+2$	$2a^2+a$	$2a^2+2a+1$
$a+1$	a^2+2a+1	a^2+a+1	a^2+2a	$2a^2+a+1$	$a+2$
$a+2$	a^2+a+1	a^2+a+2	a	$2a^2+a+2$	a^2+2
$2a$	a^2	a^2+2a	$2a^2+2a+1$	$2a^2+2a$	$2a^2+2$
$2a+1$	a^2+a+1	a^2+2a+1	a^2+2	$2a^2+2a+1$	a
$2a+2$	a^2+2a+1	a^2+2a+2	$a+2$	$2a^2+2a+2$	a^2+2a

.]

19. Use your table to list the squares in this construction of $GF(27)$.
[Answer: $0, 1, x, x{+}1, x{+}2, x^2, x^2{+}2, x^2{+}x, x^2{+}x{+}1, x^2{+}2x, x^2{+}2x{+}1, 2x^2{+}2, 2x^2{+}x{+}1, 2x^2{+}2x{+}1$.]

6.3 Finite Fields of Order 5^n.

1. Make a character table for $GF(5)$.
[Answer:

	0	1	2	3	4
0	0	1	−1	−1	1
1	1	−1	−1	1	0
2	−1	−1	1	0	1
3	−1	1	0	1	−1
4	1	0	1	−1	−1

.]

2. Use this character table to draw the Paley five graph.
[Answer: See figure 6.5.]

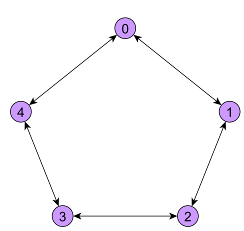


Figure 6.5: Paley Graph on Five Vertices

3. For the finite field constructed from the irreducible $x^2 + 2$ in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.

[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	3	$2a$	2	$3a$	2	$3a$	3
1	1	$a+1$	$2a+4$	$2a+1$	$4a+3$	$3a+1$	$a+3$	$3a+1$	$3a+4$
2	4	$a+2$	$4a+2$	$2a+2$	$3a+1$	$3a+2$	$2a+1$	$3a+2$	$a+2$
3	4	$a+3$	$a+2$	$2a+3$	$2a+1$	$3a+3$	$3a+1$	$3a+3$	$4a+2$
4	1	$a+4$	$3a+4$	$2a+4$	$a+3$	$3a+4$	$4a+3$	$3a+4$	$2a+4$

.]

4. List all the squares in this construction.
   [Answer: $0, 1, 2, 3, 4, a+2, a+3, 2a+1, 2a+4, 3a+1, 3a+4, 4a+2, 4a+3$.]

5. For the finite field constructed from the irreducible x^2+3 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
   [Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	2	$2a$	3	$3a$	3	$3a$	2
1	1	$a+1$	$2a+3$	$2a+1$	$4a+4$	$3a+1$	$a+4$	$3a+1$	$3a+3$
2	4	$a+2$	$4a+1$	$2a+2$	$3a+2$	$3a+2$	$2a+2$	$3a+2$	$a+1$
3	4	$a+3$	$a+1$	$2a+3$	$2a+2$	$3a+3$	$3a+2$	$3a+3$	$4a+1$
4	1	$a+4$	$3a+3$	$2a+4$	$a+4$	$3a+4$	$4a+4$	$3a+4$	$2a+3$

.]

6. List all the squares in this construction.
   [Answer: $0, 1, 2, 3, 4, a+1, a+4, 2a+2, 2a+3, 3a+2, 3a+3, 4a+1, 4a+4$.]

7. For the finite field constructed from the irreducible x^2+x+1 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
   [Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$4a+4$	$2a$	$a+1$	$3a$	$a+1$	$4a$	$4a+4$
1	1	$a+1$	a	$2a+1$	2	$3a+1$	$2a+2$	$4a+1$	$2a$
2	4	$a+2$	$3a+3$	$2a+2$	$4a$	$3a+2$	$3a$	$4a+2$	3
3	4	$a+3$	3	$2a+3$	$3a$	$3a+3$	$4a$	$4a+3$	$3a+3$
4	1	$a+4$	$2a$	$2a+4$	$2a+2$	$3a+4$	2	$4a+4$	a

.]

8. List all the squares in this construction.
   [Answer: $0, 1, 2, 3, 4, a, a+1, 2a, 2a+2, 3a, 3a+3, 4a, 4a+4$.]

9. For the finite field constructed from the irreducible x^2+x+2 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
   [Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$4a+3$	$2a$	$a+2$	$3a$	$a+2$	$4a$	$4a+3$
1	1	$a+1$	$a+4$	$2a+1$	3	$3a+1$	$2a+3$	$4a+1$	$2a+4$
2	4	$a+2$	$3a+2$	$2a+2$	$4a+1$	$3a+2$	$3a+1$	$4a+2$	2
3	4	$a+3$	2	$2a+3$	$3a+1$	$3a+3$	$4a+1$	$4a+3$	$3a+2$
4	1	$a+4$	$2a+4$	$2a+4$	$2a+3$	$3a+4$	3	$4a+4$	$a+4$

.]

10. List all the squares in this construction.
[Answer: $0, 1, a+2, a+4, 2, 2a+3, 2a+4, 3, 3a+1, 3a+2, 4, 4a+1, 4a+3$.]

11. For the finite field constructed from the irreducible x^2+2x+3 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$3a+2$	$2a$	$2a+3$	$3a$	$2a+3$	$4a$	$3a+2$
1	1	$a+1$	3	$2a+1$	$a+4$	$3a+1$	$3a+4$	$4a+1$	$a+3$
2	4	$a+2$	$2a+1$	$2a+2$	2	$3a+2$	$4a+2$	$4a+2$	$4a+1$
3	4	$a+3$	$4a+1$	$2a+3$	$4a+2$	$3a+3$	2	$4a+3$	$2a+1$
4	1	$a+4$	$a+3$	$2a+4$	$3a+4$	$3a+4$	$a+4$	$4a+4$	3

.]

12. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4a+3, a+4, 2a+1, 2a+3, 3a+2, 3a+4, 4a+1, 4a+2$.]

13. For the finite field constructed from the irreducible x^2+2x+4 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$3a+1$	$2a$	$2a+4$	$3a$	$2a+4$	$4a$	$3a+1$
1	1	$a+1$	2	$2a+1$	a	$3a+1$	$3a$	$4a+1$	$a+2$
2	4	$a+2$	$2a$	$2a+2$	3	$3a+2$	$4a+3$	$4a+2$	$4a$
3	4	$a+3$	$4a$	$2a+3$	$4a+3$	$3a+3$	3	$4a+3$	$2a$
4	1	$a+4$	$a+2$	$2a+4$	$3a$	$3a+4$	a	$4a+4$	2

.]

14. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, a, a+2, 2a, 2a+4, 3a, 3a+1, 4a, 4a+3$.]

15. For the finite field constructed from the irreducible x^2+3x+3 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$2a+2$	$2a$	$3a+3$	$3a$	$3a+3$	$4a$	$2a+2$
1	1	$a+1$	$4a+3$	$2a+1$	$2a+4$	$3a+1$	$4a+4$	$4a+1$	3
2	4	$a+2$	$a+1$	$2a+2$	$a+2$	$3a+2$	2	$4a+2$	$3a+1$
3	4	$a+3$	$3a+1$	$2a+3$	2	$3a+3$	$a+2$	$4a+3$	$a+1$
4	1	$a+4$	3	$2a+4$	$4a+4$	$3a+4$	$2a+4$	$4a+4$	$4a+3$

.]

16. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, a+1, a+2, 2a+2, 2a+4, 3a+1, 3a+3, 4a+3, 4a+4$.]

17. For the finite field constructed from the irreducible x^2+3x+4 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$2a+1$	$2a$	$3a+4$	$3a$	$3a+4$	$4a$	$2a+1$
1	1	$a+1$	$4a+2$	$2a+1$	$2a$	$3a+1$	$4a$	$4a+1$	2
2	4	$a+2$	a	$2a+2$	$a+3$	$3a+2$	3	$4a+2$	$3a$
3	4	$a+3$	$3a$	$2a+3$	3	$3a+3$	$a+3$	$4a+3$	a
4	1	$a+4$	2	$2a+4$	$4a$	$3a+4$	$2a$	$4a+4$	$4a+2$

.]

18. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, a, a+3, 2a, 2a+1, 3a, 3a+4, 4a, 4a+2$.]

19. For the finite field constructed from the irreducible x^2+4x+1 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$a+4$	$2a$	$4a+1$	$3a$	$4a+1$	$4a$	$a+4$
1	1	$a+1$	$3a$	$2a+1$	$3a+2$	$3a+1$	2	$4a+1$	$4a$
2	4	$a+2$	3	$2a+2$	$2a$	$3a+2$	a	$4a+2$	$2a+3$
3	4	$a+3$	$2a+3$	$2a+3$	a	$3a+3$	$2a$	$4a+3$	3
4	1	$a+4$	$4a$	$2a+4$	2	$3a+4$	$3a+2$	$4a+4$	$3a$

.]

20. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, a, a+4, 2a, 2a+3, 3a, 3a+2, 4a, 4a+1$.]

21. For the finite field constructed from the irreducible x^2+4x+2 in $\mathbb{Z}_5[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	$a+3$	$2a$	$4a+2$	$3a$	$4a+2$	$4a$	$a+3$
1	1	$a+1$	$3a+4$	$2a+1$	$3a+3$	$3a+1$	3	$4a+1$	$4a+4$
2	4	$a+2$	2	$2a+2$	$2a+1$	$3a+2$	$a+1$	$4a+2$	$2a+2$
3	4	$a+3$	$2a+2$	$2a+3$	$a+1$	$3a+3$	$2a+1$	$4a+3$	2
4	1	$a+4$	$4a+4$	$2a+4$	3	$3a+4$	$3a+3$	$4a+4$	$3a+4$

.]

22. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4a+1, a+3, 2a+1, 2a+2, 3a+3, 3a+4, 4a+2, 4a+4$.]

6.4 Finite Fields of Order 7^n.

1. Make a character table for $GF(7)$.
   [Answer:

	0	1	2	3	4	5	6
0	0	1	1	-1	1	-1	-1
1	-1	0	1	1	-1	1	-1
2	-1	-1	0	1	1	-1	1
3	1	-1	-1	0	1	1	-1
4	-1	1	-1	-1	0	1	1
5	1	-1	1	-1	-1	0	1
6	1	1	-1	1	-1	-1	0

.]

2. Use this character table to draw the Paley seven graph.
   [Answer: See figure 6.6.]

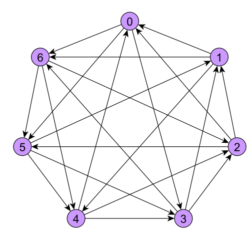


Figure 6.6: Paley Graph on Seven Vertices

3. For the finite field constructed from the irreducible x^2+1 in $\mathbb{Z}_7[x]$ make a list of squares for each of the elements.

x	x^2
0	0
1	1
2	4
3	2
4	2
5	4
6	1

x	x^2
a	6
$a+1$	$2a$
$a+2$	$4a+3$
$a+3$	$6a+1$
$a+4$	$a+1$
$a+5$	$3a+3$
$a+6$	$5a$

x	x^2
$2a$	3
$2a+1$	$4a+4$
$2a+2$	a
$2a+3$	$5a+5$
$2a+4$	$2a+5$
$2a+5$	$6a$
$2a+6$	$3a+4$

x	x^2
$3a$	5
$3a+1$	$6a+6$
$3a+2$	$5a+2$
$3a+3$	$4a$
$3a+4$	$3a$
$3a+5$	$2a+2$
$3a+6$	$a+6$

x	x^2
$4a$	5
$4a+1$	$a+6$
$4a+2$	$2a+2$
$4a+3$	$3a$
$4a+4$	$4a$
$4a+5$	$5a+2$
$4a+6$	$6a+6$

x	x^2
$5a$	3
$5a+1$	$3a+4$
$5a+2$	$6a$
$5a+3$	$2a+5$
$5a+4$	$5a+5$
$5a+5$	a
$5a+6$	$4a+4$

x	x^2
$6a$	6
$6a+1$	$5a$
$6a+2$	$3a+3$
$6a+3$	$a+1$
$6a+4$	$6a+1$
$6a+5$	$4a+3$
$6a+6$	$2a$

4. List all the squares in this construction.
[Answer: $0, 1, a, a+1, a+6, 2, 2a, 2a+2, 2a+5, 3, 3a, 3a+3, 3a+4, 4, 4a, 4a+3, 4a+4, 5, 5a, 5a+2, 5a+5, 6, 6a, 6a+1, 6a+6$.]

5. For the finite field constructed from the irreducible x^2+2 in $\mathbb{Z}_7[x]$ make a list of squares for each of the elements.

x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	5	$2a$	6	$3a$	3
1	1	$a+1$	$2a+6$	$2a+1$	$4a$	$3a+1$	$6a+4$
2	4	$a+2$	$4a+2$	$2a+2$	$a+3$	$3a+2$	$5a$
3	2	$a+3$	$6a$	$2a+3$	$5a+1$	$3a+3$	$4a+5$
4	2	$a+4$	a	$2a+4$	$2a+1$	$3a+4$	$3a+5$
5	4	$a+5$	$3a+2$	$2a+5$	$6a+3$	$3a+5$	$2a$
6	1	$a+6$	$5a+6$	$2a+6$	$3a$	$3a+6$	$a+4$

x	x^2	x	x^2	x	x^2
$4a$	3	$5a$	6	$6a$	5
$4a+1$	$a+4$	$5a+1$	$3a$	$6a+1$	$5a+6$
$4a+2$	$2a$	$5a+2$	$6a+3$	$6a+2$	$3a+2$
$4a+3$	$3a+5$	$5a+3$	$2a+1$	$6a+3$	a
$4a+4$	$4a+5$	$5a+4$	$5a+1$	$6a+4$	$6a$
$4a+5$	$5a$	$5a+5$	$a+3$	$6a+5$	$4a+2$
$4a+6$	$6a+4$	$5a+6$	$4a$	$6a+6$	$2a+6$

6. List all the squares in this construction.
[Answer:$0, 1, a, a+3, a+4, 2, 2a, 2a+1, 2a+6, 3, 3a, 3a+2, 3a+5, 4, 4a, 4a+2, 4a+5, 5, 5a, 5a+1, 5a+6, 6, 6a, 6a+3, 6a+4$.]

7. For the finite field constructed from the irreducible x^2+4 in $\mathbb{Z}_7[x]$ make a list of squares for each of the elements.

x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	3	$2a$	5	$3a$	6
1	1	$a+1$	$2a+4$	$2a+1$	$4a+6$	$3a+1$	$6a$
2	4	$a+2$	$4a$	$2a+2$	$a+2$	$3a+2$	$5a+3$
3	2	$a+3$	$6a+5$	$2a+3$	$5a$	$3a+3$	$4a+1$
4	2	$a+4$	$a+5$	$2a+4$	$2a$	$3a+4$	$3a+1$
5	4	$a+5$	$3a$	$2a+5$	$6a+2$	$3a+5$	$2a+3$
6	1	$a+6$	$5a+4$	$2a+6$	$3a+6$	$3a+6$	a

x	x^2	x	x^2	x	x^2
$4a$	6	$5a$	5	$6a$	3
$4a+1$	a	$5a+1$	$3a+6$	$6a+1$	$5a+4$
$4a+2$	$2a+3$	$5a+2$	$6a+2$	$6a+2$	$3a$
$4a+3$	$3a+1$	$5a+3$	$2a$	$6a+3$	$a+5$
$4a+4$	$4a+1$	$5a+4$	$5a$	$6a+4$	$6a+5$
$4a+5$	$5a+3$	$5a+5$	$a+2$	$6a+5$	$4a$
$4a+6$	$6a$	$5a+6$	$4a+6$	$6a+6$	$2a+4$

8. List all the squares in this construction.
[Answer: $0, 1, a, a+2, a+5, 2, 2a, 2a+3, 2a+4, 3, 3a, 3a+1, 3a+6, 4, 4a, 4a+1, 4a+6, 5, 5a, 5a+3, 5a+4, 6, 6a, 6a+2, 6a+5$.]

6.5 Finite Fields of Order 11^n.

1. Make a character table for $GF(11)$.
[Answer:

	0	1	2	3	4	5	6	7	8	9	10
0	0	−1	1	−1	−1	−1	1	1	1	−1	1
1	1	0	−1	1	−1	−1	−1	1	1	1	−1
2	−1	1	0	−1	1	−1	−1	−1	1	1	1
3	1	−1	1	0	−1	1	−1	−1	−1	1	1
4	1	1	−1	1	0	−1	1	−1	−1	−1	1
5	1	1	1	−1	1	0	−1	1	−1	−1	−1
6	−1	1	1	1	−1	1	0	−1	1	−1	−1
7	−1	−1	1	1	1	−1	1	0	−1	1	−1
8	−1	−1	−1	1	1	1	−1	1	0	−1	1
9	1	−1	−1	−1	1	1	1	−1	1	0	−1
10	−1	1	−1	−1	−1	1	1	1	−1	1	0

.]

2. Use this character table to draw the Paley eleven graph.
[Answer: See figure 6.7.]

3. For the finite field constructed from the irreducible x^2+1 in $\mathbb{Z}_{11}[x]$ find a formula for the square of $ba+c$.
[Answer: Here $x^2 = -1$ so $(ab+c)^2 = -b^2 + 2bca + c^2 = (2bc)a + (c^2 - b^2)$.]

4. For the finite field constructed from the irreducible x^2+1 in $\mathbb{Z}_{11}[x]$ make a list of squares for each of the elements.[3]

[3] The author is aware that each of these types of problems contains 121 different problems, however there are many tricks one can use to cut this number down a bit, such as using the fact that $(-x)^2 = x^2$ or plugging into the formula we've already found.

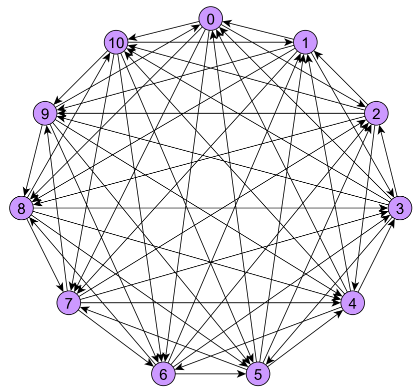


Figure 6.7: Paley Graph on Eleven Vertices

[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	10	$2a$	7	$3a$	2	$4a$	6	$5a$	8
1	1	$a+1$	$2a$	$2a+1$	$4a+8$	$3a+1$	$6a+3$	$4a+1$	$8a+7$	$5a+1$	$10a+9$
2	4	$a+2$	$4a+3$	$2a+2$	$8a$	$3a+2$	$a+6$	$4a+2$	$5a+10$	$5a+2$	$9a+1$
3	9	$a+3$	$6a+8$	$2a+3$	$a+5$	$3a+3$	$7a$	$4a+3$	$2a+4$	$5a+3$	$8a+6$
4	5	$a+4$	$8a+4$	$2a+4$	$5a+1$	$3a+4$	$2a+7$	$4a+4$	$10a$	$5a+4$	$7a+2$
5	3	$a+5$	$10a+2$	$2a+5$	$9a+10$	$3a+5$	$8a+5$	$4a+5$	$7a+9$	$5a+5$	$6a$
6	3	$a+6$	$a+2$	$2a+6$	$2a+10$	$3a+6$	$3a+5$	$4a+6$	$4a+9$	$5a+6$	$5a$
7	5	$a+7$	$3a+4$	$2a+7$	$6a+1$	$3a+7$	$9a+7$	$4a+7$	a	$5a+7$	$4a+2$
8	9	$a+8$	$5a+8$	$2a+8$	$10a+5$	$3a+8$	$4a$	$4a+8$	$9a+4$	$5a+8$	$3a+6$
9	4	$a+9$	$7a+3$	$2a+9$	$3a$	$3a+9$	$10a+6$	$4a+9$	$6a+10$	$5a+9$	$2a+1$
10	1	$a+10$	$9a$	$2a+10$	$7a+8$	$3a+10$	$5a+3$	$4a+10$	$3a+7$	$5a+10$	$a+9$

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
$6a$	8	$7a$	6	$8a$	2	$9a$	7	$10a$	10
$6a+1$	$a+9$	$7a+1$	$3a+7$	$8a+1$	$5a+3$	$9a+1$	$7a+8$	$10a+1$	$9a$
$6a+2$	$2a+1$	$7a+2$	$6a+10$	$8a+2$	$10a+6$	$9a+2$	$3a$	$10a+2$	$7a+3$
$6a+3$	$3a+6$	$7a+3$	$9a+4$	$8a+3$	$4a$	$9a+3$	$10a+5$	$10a+3$	$5a+8$
$6a+4$	$4a+2$	$7a+4$	a	$8a+4$	$9a+7$	$9a+4$	$6a+1$	$10a+4$	$3a+4$
$6a+5$	$5a$	$7a+5$	$4a+9$	$8a+5$	$3a+5$	$9a+5$	$2a+10$	$10a+5$	$a+2$
$6a+6$	$6a$	$7a+6$	$7a+9$	$8a+6$	$8a+5$	$9a+6$	$9a+10$	$10a+6$	$10a+2$
$6a+7$	$7a+2$	$7a+7$	$10a$	$8a+7$	$2a+7$	$9a+7$	$5a+1$	$10a+7$	$8a+4$
$6a+8$	$8a+6$	$7a+8$	$2a+4$	$8a+8$	$7a$	$9a+8$	$a+5$	$10a+8$	$6a+8$
$6a+9$	$9a+1$	$7a+9$	$5a+10$	$8a+9$	$a+6$	$9a+9$	$8a$	$10a+9$	$4a+3$
$6a+10$	$10a+9$	$7a+10$	$8a+7$	$8a+10$	$6a+3$	$9a+10$	$4a+8$	$10a+10$	$2a$

.]

5. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, a+2, a+5, a+6, a+9, 2a, 2a+1, 2a+4, 2a+7, 2a+10, 3a, 3a+$

$4, 3a+5, 3a+6, 3a+7, 4a, 4a+2, 4a+3, 4a+8, 4a+9, 5a, 5a+1, 5a+3, 5a+8, 5a+10, 6a, 6a+1, 6a+3, 6a+8, 6a+10, 7a, 7a+2, 7a+3, 7a+8, 7a+9, 8a, 8a+4, 8a+5, 8a+6, 8a+7, 9a, 9a+1, 9a+4, 9a+7, 9a+10, 110a, 10a+2, 10a+5, 10a+6, 10a+9.$]

6. For the finite field constructed from the irreducible x^2+3 in $\mathbb{Z}_{11}[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	8	$2a$	10	$3a$	6	$4a$	7	$5a$	2
1	1	$a+1$	$2a+9$	$2a+1$	$4a$	$3a+1$	$6a+7$	$4a+1$	$8a+8$	$5a+1$	$10a+3$
2	4	$a+2$	$4a+1$	$2a+2$	$8a+3$	$3a+2$	$a+10$	$4a+2$	$5a$	$5a+2$	$9a+6$
3	9	$a+3$	$6a+6$	$2a+3$	$a+8$	$3a+3$	$7a+4$	$4a+3$	$2a+5$	$5a+3$	$8a$
4	5	$a+4$	$8a+2$	$2a+4$	$5a+4$	$3a+4$	$2a$	$4a+4$	$10a+1$	$5a+4$	$7a+7$
5	3	$a+5$	$10a$	$2a+5$	$9a+2$	$3a+5$	$8a+9$	$4a+5$	$7a+10$	$5a+5$	$6a+5$
6	3	$a+6$	a	$2a+6$	$2a+2$	$3a+6$	$3a+9$	$4a+6$	$4a+10$	$5a+6$	$5a+5$
7	5	$a+7$	$3a+2$	$2a+7$	$6a+4$	$3a+7$	$9a$	$4a+7$	$a+1$	$5a+7$	$4a+7$
8	9	$a+8$	$5a+6$	$2a+8$	$10a+8$	$3a+8$	$4a+4$	$4a+8$	$9a+5$	$5a+8$	$3a$
9	4	$a+9$	$7a+1$	$2a+9$	$3a+3$	$3a+9$	$10a+10$	$4a+9$	$6a$	$5a+9$	$2a+6$
10	1	$a+10$	$9a+9$	$2a+10$	$7a$	$3a+10$	$5a+7$	$4a+10$	$3a+8$	$5a+10$	$a+3$

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
$6a$	2	$7a$	7	$8a$	6	$9a$	10	$10a$	8
$6a+1$	$a+3$	$7a+1$	$3a+8$	$8a+1$	$5a+7$	$9a+1$	$7a$	$10a+1$	$9a+9$
$6a+2$	$2a+6$	$7a+2$	$6a$	$8a+2$	$10a+10$	$9a+2$	$3a+3$	$10a+2$	$7a+1$
$6a+3$	$3a$	$7a+3$	$9a+5$	$8a+3$	$4a+4$	$9a+3$	$10a+8$	$10a+3$	$5a+6$
$6a+4$	$4a+7$	$7a+4$	$a+1$	$8a+4$	$9a$	$9a+4$	$6a+4$	$10a+4$	$3a+2$
$6a+5$	$5a+5$	$7a+5$	$4a+10$	$8a+5$	$3a+9$	$9a+5$	$2a+2$	$10a+5$	a
$6a+6$	$6a+5$	$7a+6$	$7a+10$	$8a+6$	$8a+9$	$9a+6$	$9a+2$	$10a+6$	$10a$
$6a+7$	$7a+7$	$7a+7$	$10a+1$	$8a+7$	$2a$	$9a+7$	$5a+4$	$10a+7$	$8a+2$
$6a+8$	$8a$	$7a+8$	$2a+5$	$8a+8$	$7a+4$	$9a+8$	$a+8$	$10a+8$	$6a+6$
$6a+9$	$9a+6$	$7a+9$	$5a$	$8a+9$	$a+10$	$9a+9$	$8a+3$	$10a+9$	$4a+1$
$6a+10$	$10a+3$	$7a+10$	$8a+8$	$8a+10$	$6a+7$	$9a+10$	$4a$	$10a+10$	$2a+9$

.]

7. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, a+1, a+3, a+8, a+10, 2a, 2a+2, 2a+5, 2a+6, 2a+9, 3a, 3a+2, 3a+3, 3a+8, 3a+9, 4a, 4a+1, 4a+4, 4a+7, 4a+10, 5a, 5a+4, 5a+5, 5a+6, 5a+7, 6a, 6a+4, 6a+5, 6a+6, 6a+7, 7a, 7a+1, 7a+4, 7a+7, 7a+10, 8a, 8a+2, 8a+3, 8a+8, 8a+9, 9a, 9a+2, 9a+5, 9a+6, 9a+9, 110a, 10a+1, 10a+3, 10a+8, 10a+10.$]

8. For the finite field constructed from the irreducible x^2+4 in $\mathbb{Z}_{11}[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	7	$2a$	6	$3a$	8	$4a$	2	$5a$	10
1	1	$a+1$	$2a+8$	$2a+1$	$4a+7$	$3a+1$	$6a+9$	$4a+1$	$8a+3$	$5a+1$	$10a$
2	4	$a+2$	$4a$	$2a+2$	$8a+10$	$3a+2$	$a+1$	$4a+2$	$5a+6$	$5a+2$	$9a+3$
3	9	$a+3$	$6a+5$	$2a+3$	$a+4$	$3a+3$	$7a+6$	$4a+3$	$2a$	$5a+3$	$8a+8$
4	5	$a+4$	$8a+1$	$2a+4$	$5a$	$3a+4$	$2a+2$	$4a+4$	$10a+7$	$5a+4$	$7a+4$
5	3	$a+5$	$10a+10$	$2a+5$	$9a+9$	$3a+5$	$8a$	$4a+5$	$7a+5$	$5a+5$	$6a+2$
6	3	$a+6$	$a+10$	$2a+6$	$2a+9$	$3a+6$	$3a$	$4a+6$	$4a+5$	$5a+6$	$5a+2$
7	5	$a+7$	$3a+1$	$2a+7$	$6a$	$3a+7$	$9a+2$	$4a+7$	$a+7$	$5a+7$	$4a+4$
8	9	$a+8$	$5a+5$	$2a+8$	$10a+4$	$3a+8$	$4a+6$	$4a+8$	$9a$	$5a+8$	$3a+8$
9	4	$a+9$	$7a$	$2a+9$	$3a+10$	$3a+9$	$10a+1$	$4a+9$	$6a+6$	$5a+9$	$2a+3$
10	1	$a+10$	$9a+8$	$2a+10$	$7a+7$	$3a+10$	$5a+9$	$4a+10$	$3a+3$	$5a+10$	a

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
$6a$	10	$7a$	2	$8a$	8	$9a$	6	$10a$	7
$6a+1$	a	$7a+1$	$3a+3$	$8a+1$	$5a+9$	$9a+1$	$7a+7$	$10a+1$	$9a+8$
$6a+2$	$2a+3$	$7a+2$	$6a+6$	$8a+2$	$10a+1$	$9a+2$	$3a+10$	$10a+2$	$7a$
$6a+3$	$3a+8$	$7a+3$	$9a$	$8a+3$	$4a+6$	$9a+3$	$10a+4$	$10a+3$	$5a+5$
$6a+4$	$4a+4$	$7a+4$	$a+7$	$8a+4$	$9a+2$	$9a+4$	$6a$	$10a+4$	$3a+1$
$6a+5$	$5a+2$	$7a+5$	$4a+5$	$8a+5$	$3a$	$9a+5$	$2a+9$	$10a+5$	$a+10$
$6a+6$	$6a+2$	$7a+6$	$7a+5$	$8a+6$	$8a$	$9a+6$	$9a+9$	$10a+6$	$10a+10$
$6a+7$	$7a+4$	$7a+7$	$10a+7$	$8a+7$	$2a+2$	$9a+7$	$5a$	$10a+7$	$8a+1$
$6a+8$	$8a+8$	$7a+8$	$2a$	$8a+8$	$7a+6$	$9a+8$	$a+4$	$10a+8$	$6a+5$
$6a+9$	$9a+3$	$7a+9$	$5a+6$	$8a+9$	$a+1$	$9a+9$	$8a+10$	$10a+9$	$4a$
$6a+10$	$10a$	$7a+10$	$8a+3$	$8a+10$	$6a+9$	$9a+10$	$4a+7$	$10a+10$	$2a+8$

.]

9. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, a+1, a+4, a+7, a+10, 2a, 2a+2, 2a+3, 2a+8, 2a+9, 3a, 3a+1, 3a+3, 3a+8, 3a+10, 4a, 4a+4, 4a+5, 4a+6, 4a+7, 5a, 5a+2, 5a+5, 5a+6, 5a+9, 6a, 6a+2, 6a+5, 6a+6, 6a+9, 7a, 7a+4, 7a+5, 7a+6, 7a+7, 8a, 8a+1, 8a+3, 8a+8, 8a+10, 9a, 9a+2, 9a+3, 9a+8, 9a+9, 110a, 10a+1, 10a+4, 10a+7, 10a+10.$]

10. For the finite field constructed from the irreducible x^2+5 in $\mathbb{Z}_{11}[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	6	$2a$	2	$3a$	10	$4a$	8	$5a$	7
1	1	$a+1$	$2a+7$	$2a+1$	$4a+3$	$3a+1$	$6a$	$4a+1$	$8a+9$	$5a+1$	$10a+8$
2	4	$a+2$	$4a+10$	$2a+2$	$8a+6$	$3a+2$	$a+3$	$4a+2$	$5a+1$	$5a+2$	$9a$
3	9	$a+3$	$6a+4$	$2a+3$	a	$3a+3$	$7a+8$	$4a+3$	$2a+6$	$5a+3$	$8a+5$
4	5	$a+4$	$8a$	$2a+4$	$5a+7$	$3a+4$	$2a+4$	$4a+4$	$10a+2$	$5a+4$	$7a+1$
5	3	$a+5$	$10a+9$	$2a+5$	$9a+5$	$3a+5$	$8a+2$	$4a+5$	$7a$	$5a+5$	$6a+10$
6	3	$a+6$	$a+9$	$2a+6$	$2a+5$	$3a+6$	$3a+2$	$4a+6$	$4a$	$5a+6$	$5a+10$
7	5	$a+7$	$3a$	$2a+7$	$6a+7$	$3a+7$	$9a+4$	$4a+7$	$a+2$	$5a+7$	$4a+1$
8	9	$a+8$	$5a+4$	$2a+8$	$10a$	$3a+8$	$4a+8$	$4a+8$	$9a+6$	$5a+8$	$3a+5$
9	4	$a+9$	$7a+10$	$2a+9$	$3a+6$	$3a+9$	$10a+3$	$4a+9$	$6a+1$	$5a+9$	$2a$
10	1	$a+10$	$9a+7$	$2a+10$	$7a+3$	$3a+10$	$5a$	$4a+10$	$3a+9$	$5a+10$	$a+8$

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
$6a$	7	$7a$	8	$8a$	10	$9a$	2	$10a$	6
$6a+1$	$a+8$	$7a+1$	$3a+9$	$8a+1$	$5a$	$9a+1$	$7a+3$	$10a+1$	$9a+7$
$6a+2$	$2a$	$7a+2$	$6a+1$	$8a+2$	$10a+3$	$9a+2$	$3a+6$	$10a+2$	$7a+10$
$6a+3$	$3a+5$	$7a+3$	$9a+6$	$8a+3$	$4a+8$	$9a+3$	$10a$	$10a+3$	$5a+4$
$6a+4$	$4a+1$	$7a+4$	$a+2$	$8a+4$	$9a+4$	$9a+4$	$6a+7$	$10a+4$	$3a$
$6a+5$	$5a+10$	$7a+5$	$4a$	$8a+5$	$3a+2$	$9a+5$	$2a+5$	$10a+5$	$a+9$
$6a+6$	$6a+10$	$7a+6$	$7a$	$8a+6$	$8a+2$	$9a+6$	$9a+5$	$10a+6$	$10a+9$
$6a+7$	$7a+1$	$7a+7$	$10a+2$	$8a+7$	$2a+4$	$9a+7$	$5a+7$	$10a+7$	$8a$
$6a+8$	$8a+5$	$7a+8$	$2a+6$	$8a+8$	$7a+8$	$9a+8$	a	$10a+8$	$6a+4$
$6a+9$	$9a$	$7a+9$	$5a+1$	$8a+9$	$a+3$	$9a+9$	$8a+6$	$10a+9$	$4a+10$
$6a+10$	$10a+8$	$7a+10$	$8a+9$	$8a+10$	$6a$	$9a+10$	$4a+3$	$10a+10$	$2a+7$

.]

11. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, a+2, a+3, a+8, a+9, 2a, 2a+4, 2a+5, 2a+6, 2a+7, 3a, 3a+2, 3a+5, 3a+6, 3a+9, 4a, 4a+1, 4a+3, 4a+8, 4a+10, 5a, 5a+1, 5a+4, 5a+7, 5a+10, 6a, 6a+1, 6a+4, 6a+7, 6a+10, 7a, 7a+1, 7a+3, 7a+8, 7a+10, 8a, 8a+2, 8a+5, 8a+6, 8a+9, 9a, 9a+4, 9a+5, 9a+6, 9a+7, 110a, 10a+2, 10a+3, 10a+8, 10a+9$.]

12. For the finite field constructed from the irreducible $x^2 + 9$ in $\mathbb{Z}_{11}[x]$ make a list of squares for each of the elements.
[Answer:

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
0	0	a	2	$2a$	8	$3a$	7	$4a$	10	$5a$	6
1	1	$a+1$	$2a+3$	$2a+1$	$4a+9$	$3a+1$	$6a+8$	$4a+1$	$8a$	$5a+1$	$10a+7$
2	4	$a+2$	$4a+6$	$2a+2$	$8a+1$	$3a+2$	a	$4a+2$	$5a+3$	$5a+2$	$9a+10$
3	9	$a+3$	$6a$	$2a+3$	$a+6$	$3a+3$	$7a+5$	$4a+3$	$2a+8$	$5a+3$	$8a+4$
4	5	$a+4$	$8a+7$	$2a+4$	$5a+2$	$3a+4$	$2a+1$	$4a+4$	$10a+4$	$5a+4$	$7a$
5	3	$a+5$	$10a+5$	$2a+5$	$9a$	$3a+5$	$8a+10$	$4a+5$	$7a+2$	$5a+5$	$6a+9$
6	3	$a+6$	$a+5$	$2a+6$	$2a$	$3a+6$	$3a+10$	$4a+6$	$4a+2$	$5a+6$	$5a+9$
7	5	$a+7$	$3a+7$	$2a+7$	$6a+2$	$3a+7$	$9a+1$	$4a+7$	$a+4$	$5a+7$	$4a$
8	9	$a+8$	$5a$	$2a+8$	$10a+6$	$3a+8$	$4a+5$	$4a+8$	$9a+8$	$5a+8$	$3a+4$
9	4	$a+9$	$7a+6$	$2a+9$	$3a+1$	$3a+9$	$10a$	$4a+9$	$6a+3$	$5a+9$	$2a+10$
10	1	$a+10$	$9a+3$	$2a+10$	$7a+9$	$3a+10$	$5a+8$	$4a+10$	$3a$	$5a+10$	$a+7$

x	x^2	x	x^2	x	x^2	x	x^2	x	x^2
$6a$	6	$7a$	10	$8a$	7	$9a$	8	$10a$	2
$6a+1$	$a+7$	$7a+1$	$3a$	$8a+1$	$5a+8$	$9a+1$	$7a+9$	$10a+1$	$9a+3$
$6a+2$	$2a+10$	$7a+2$	$6a+3$	$8a+2$	$10a$	$9a+2$	$3a+1$	$10a+2$	$7a+6$
$6a+3$	$3a+4$	$7a+3$	$9a+8$	$8a+3$	$4a+5$	$9a+3$	$10a+6$	$10a+3$	$5a$
$6a+4$	$4a$	$7a+4$	$a+4$	$8a+4$	$9a+1$	$9a+4$	$6a+2$	$10a+4$	$3a+7$
$6a+5$	$5a+9$	$7a+5$	$4a+2$	$8a+5$	$3a+10$	$9a+5$	$2a$	$10a+5$	$a+5$
$6a+6$	$6a+9$	$7a+6$	$7a+2$	$8a+6$	$8a+10$	$9a+6$	$9a$	$10a+6$	$10a+5$
$6a+7$	$7a$	$7a+7$	$10a+4$	$8a+7$	$2a+1$	$9a+7$	$5a+2$	$10a+7$	$8a+7$
$6a+8$	$8a+4$	$7a+8$	$2a+8$	$8a+8$	$7a+5$	$9a+8$	$a+6$	$10a+8$	$6a$
$6a+9$	$9a+10$	$7a+9$	$5a+3$	$8a+9$	a	$9a+9$	$8a+1$	$10a+9$	$4a+6$
$6a+10$	$10a+7$	$7a+10$	$8a$	$8a+10$	$6a+8$	$9a+10$	$4a+9$	$10a+10$	$2a+3$

.]

13. List all the squares in this construction.
[Answer: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, a, a+4, a+5, a+6, a+7, 2a, 2a+1, 2a+3, 2a+8, 2a+10, 3a, 3a+1, 3a+4, 3a+7, 3a+10, 4a, 4a+2, 4a+5, 4a+6, 4a+9, 5a, 5a+2, 5a+3, 5a+8, 5a+9, 6a, 6a+2, 6a+3, 6a+8, 6a+9, 7a, 7a+2, 7a+5, 7a+6, 7a+9, 8a, 8a+1, 8a+4, 8a+7, 8a+10, 9a, 9a+1, 9a+3, 9a+8, 9a+10, 110a, 10a+4, 10a+5, 10a+6, 10a+7$.]

Chapter 7

Linear Codes

7.1 Linear $[n, k, d]$-Codes and Duals

A generator matrix is given for a linear code. Find the values of n, d and k for which this matrix generates an $[n, k, d]-$ code. Find the number of errors this code can detect, and the number of errors this code can correct. Find a this information for the dual code as well and produce a generator matrix for this dual. State whether the code is self-dual. Note that since generator matrices are not unique, answers may look different[1] and it may not immediately be obvious if a code is self-dual. Feel free to use the fact that when $G = [I|P]$, then $H = [-P^T|I]$ whenever applicable.

1. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 \end{bmatrix}$
   [Answer: $[2, 1, 2]-$code, Detects 1, Corrects 0, G is self dual.]

2. $|F| = 2$, $G = \begin{bmatrix} 1 & 0 \end{bmatrix}$
   [Answer: $[2, 1, 1]-$code, Detects 0, Corrects 0, $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ generates a $[2, 1, 1]-$code, Detects 0, Corrects 0.]

3. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
   [Answer: $[3, 1, 3]-$code, Detects 2, Corrects 1, $H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ generates a $[3, 2, 2]-$code, Detects 1, Corrects 0.]

4. $|F| = 2$, $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ This is a punctured Hadamard code.
   [Answer: $[3, 2, 2]-$code, Detects 1, Corrects 0, $H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ generates a $[3, 1, 3]-$code, Detects 2, Corrects 1.]

5. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$
   [Answer: $[3, 2, 1]-$code, Detects 0, Corrects 0, $H = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ generates a $[3, 1, 2]-$code, Detects 1, Corrects 0.]

[1] Though the values of n,k and d will remain the same.

6. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

   [Answer: $[4,1,4]-$code, Detects 3, Corrects 1 $H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ generates a $[4,3,2]-$code, Detects 1, Corrects 0.]

7. $|F| = 2$, $G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

   [Answer: $[4,2,2]-$code, Detects 1, Corrects 0, G is self dual.]

8. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

   [Answer: $[4,2,2]-$code, Detects 1, Corrects 0, G is self dual.]

9. $|F| = 2$, $G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

   [Answer: $[4,2,2]-$code, Detects 1, Corrects 0, $H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ generates a $[4,2,2]-$code, Detects 1, Corrects 0.]

10. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

    [Answer: $[5,1,5]-$code, Detects 4, Corrects 2, $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ generates a $[5,4,2]-$code, Detects 1, Corrects 0.]

11. $|F| = 2$, $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

    [Answer: $[5,3,2]-$code, Detects 1, Corrects 0, $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ generates a $[5,2,3]-$code, Detects 2, Corrects 1.]

12. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

    [Answer: $[5,2,2]-$code, Detects 1, Corrects 0, $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$ generates a $[5,2,2]-$code, Detects 1, Corrects 0.]

13. $|F| = 2$, $G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

    [Answer: $[6,2,3]-$code, Detects 2, Corrects 1, $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ generates a $[6,2,2]-$code, Detects 1, Corrects 0.]

14. $|F|=2$, $G=\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix}$

[Answer: $[6,3,3]-$code, Detects 2, Corrects 1, $H=\begin{bmatrix}1&1&0&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{bmatrix}$ generates a $[6,3,3]-$code, Detects 2, Corrects 1.]

15. $|F|=2$, $G=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&1&0&1\\0&0&1&1&1&0\end{bmatrix}$

[Answer: $[6,3,3]-$code, Detects 2, Corrects 1, $H=\begin{bmatrix}0&1&1&1&0&0\\1&0&1&0&1&0\\1&1&0&0&0&1\end{bmatrix}$ generates a $[6,3,3]-$code, Detects 2, Corrects 1.]

16. $|F|=2$, $G=\begin{bmatrix}1&0&1&1&0&1\\0&1&1&0&1&1\end{bmatrix}$

[Answer: $[6,2,4]-$code, Detects 3, Corrects 1. $H=\begin{bmatrix}1&0&0&0&1&1\\0&1&0&0&1&0\\0&0&1&0&0&1\\0&0&0&1&1&1\end{bmatrix}$ generates a $[6,4,2]-$code, Detects 1, Corrects 0.]

17. $|F|=2$, $G=\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}$ This is a punctured Hadamard code.

[Answer: $[7,3,4]-$code, Detects 3, Corrects 1, $H=\begin{bmatrix}1&0&0&0&0&1&1\\0&1&0&0&1&0&1\\0&0&1&0&1&1&0\\0&0&0&1&1&1&1\end{bmatrix}$ generates a $[7,4,3]-$code, Detects 2, Corrects 1.]

18. $|F|=2$, $G=\begin{bmatrix}1&0&0&0&1&1&0\\0&1&0&0&1&0&1\\0&0&1&0&0&1&1\\0&0&0&1&1&1&1\end{bmatrix}$ This is the generator matrix for the [7,4] Hamming code.

[Answer: $[7,4,3]-$code, Detects 2, Corrects 1, $H=\begin{bmatrix}1&1&0&1&1&0&0\\1&0&1&1&0&1&0\\0&1&1&1&0&0&1\end{bmatrix}$ generates a $[7,3,4]-$code, Detects 3, Corrects 1.]

19. $|F|=2$, $G=\begin{bmatrix}1&0&0&0&0&1&1&1\\0&1&0&0&1&0&1&1\\0&0&1&0&1&1&0&1\\0&0&0&1&1&1&1&0\end{bmatrix}$. This is the generator matrix for what we call the [8,4] extended Hamming code.
[Answer: $[8,4,4]-$code, Detects 3, Corrects 1. This code is self dual.]

20. $|F| = 3$, $G = \begin{bmatrix} 1 & 1 \end{bmatrix}$
[Answer: $[2,1,2]-$code, Detects 1, Corrects 0. $H = \begin{bmatrix} 1 & 2 \end{bmatrix}$ generates a $[2,1,2]-$code, Detects 1, Corrects 0.]

21. $|F| = 3$, $G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
[Answer: $[3,1,3]-$code, Detects 2, Corrects 1. $H = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ generates a $[3,2,2]-$code, Detects 1, Corrects 0.]

22. $|F| = 3$, $G = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$
[Answer: $[3,1,3]-$code, Detects 2, Corrects 1. $H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ generates a $[3,2,2]-$code, Detects 1, Corrects 0.]

23. $|F| = 3$, $G = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$
[Answer: $[4,1,4]-$code, Detects 3, Corrects 1. $H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ generates a $[4,3,2]-$code, Detects 1, Corrects 0.]

24. $|F| = 3$, $G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$
[Answer: $[4,2,2]-$code, Detects 1, Corrects 0. $H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ generates a $[4,2,2]-$code, Detects 1, Corrects 0.]

25. $|F| = 3$, $G = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$
[Answer: $[4,2,2]-$code, Detects 1, Corrects 0. $H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ generates a $[4,2,2]-$code, Detects 1, Corrects 0.]

26. $|F| = 3$, $G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$
[Answer: $[4,2,2]-$code, Detects 1, Corrects 0. $H = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ generates a $[4,2,2]-$code, Detects 1, Corrects 0.]

27. $|F| = 3$, $G = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ This is the ternary [4,2]-Hamming code.
[Answer: $[4,2,3]-$code, Detects 2, Corrects 1. This code is self dual.]

28. $|F| = 3$, $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
[Answer: $[4,3,2]-$code, Detects 3, Corrects 1. $H = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$ generates a $[4,1,4]-$code, Detects 1, Corrects 0.]

29. $|F| = 3$, $G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

[Answer: $[5,1,5]-$code, Detects 4, Corrects 2. $H = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ generates a $[5,4,2]-$code, Detects 1, Corrects 0.]

30. $|F| = 3$, $G = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$

[Answer: $[5,4,1]-$code, Detects 0, Corrects 0. $H = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \end{bmatrix}$ generates a $[5,1,3]-$code, Detects 2, Corrects 1.]

31. $|F| = 3$, $G = \begin{bmatrix} 2 & 1 & 2 & 1 & 2 \end{bmatrix}$

[Answer: $[5,1,5]-$code, Detects 4, Corrects 2. $H = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ generates a $[5,4,2]-$code, Detects 1, Corrects 0.]

32. $|F| = 3$, $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$

[Answer: $[6,2,4]-$code, Detects 3, Corrects 1. $H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$ generates a $[6,4,2]-$code, Detects 1, Corrects 0.]

7.2 Hadamard Codes

Recall we have introduced several ways of looking at Hadamard codes.

- Given a binary string $x \in \mathbb{Z}_2^n$ we can encode x with the string of all $x \cdot y$ for all $x \in \mathbb{Z}_2^n$. This construction gives us the $[2^n, n]$-Hadamard code. For the punctured version we take $x \cdot y$ for only the y beginning with 1 to find the $[2^{n-1}, n]$-punctured Hadamard code.

- We create a generator matrix by taking the collection of all $y \in \mathbb{Z}_2^n$. Though the order in which we take the y affects our vectors, what we get is always equivalent.

- Using Sylvester's construction, form the $2^n \times 2^n$ matrix of codewords for the $[2^n, n]$ Hadamard code. We can find the $[2^n, n+1]$ punctured Hadamard code by taking the $2^n \times 2^n$ matrix of codewords, calling it A and forming $\begin{bmatrix} A \\ A' \end{bmatrix}$ where A' has each entry reversed.

- We can also find the generator matrix for the punctured $[2^n, n+1]$ code by taking the generator matrix for the $[2^n, n]$ Hadamard code and adding a row of ones to the top.

1. Use the inner product method to construct Hadamard $[2, 1]$-code. State what each binary string is translated into.
[Answer: After forming

x	$x \cdot [0]$	$x \cdot [1]$
$[0]$	0	0
$[1]$	0	1

we see that $[0]$ is translated as $[0, 0]$ and $[1]$ is translated as $[0, 1]$.]

2. Use the inner product method to construct the Hadamard $[4, 2]$-code. State what each binary string is translated into, then do the same for the punctured version.
[Answer: After forming

x	$x \cdot [0, 0]$	$x \cdot [0, 1]$	$x \cdot [1, 0]$	$x \cdot [1, 1]$
$[0, 0]$	0	0	0	0
$[0, 1]$	0	1	0	1
$[1, 0]$	0	0	1	1
$[1, 1]$	0	1	1	0

we can make a table for all possible codewords

x	$Had(x)$
$[0, 0]$	$[0, 0, 0, 0]$
$[0, 1]$	$[0, 1, 0, 1]$
$[1, 0]$	$[0, 0, 1, 1]$
$[1, 1]$	$[0, 1, 1, 0]$

.

For the punctured version we take the dot product of x with only the vectors beginning with one to get

x	$x \cdot [1, 0]$	$x \cdot [1, 1]$
$[0, 0]$	0	0
$[0, 1]$	0	1
$[1, 0]$	1	1
$[1, 1]$	1	0

x	$Had(x)$
$[0, 0]$	$[0, 0]$
$[0, 1]$	$[0, 1]$
$[1, 0]$	$[1, 1]$
$[1, 1]$	$[1, 0]$

.]

3. Find a generator matrix for each of these.
[Answer: We form a matrix from the rows corresponding to Had($[1, 0]$), and Had($[0, 1]$), to get $G = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. We do the same for the punctured version which gives us $G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. .]

4. Use the inner product method to construct the Hadamard $[8, 4]$-code. State what each binary string is translated into, then do the same for the punctured version.

[Answer: After forming

x	$x\cdot[0,0,0]$	$x\cdot[0,0,1]$	$x\cdot[0,1,0]$	$x\cdot[0,1,1]$	$x\cdot[1,0,0]$	$x\cdot[1,0,1]$	$x\cdot[1,1,0]$	$x\cdot[1,1,1]$
$[0,0,0]$	0	0	0	0	0	0	0	0
$[0,0,1]$	0	1	0	1	0	1	0	1
$[0,1,0]$	0	0	1	1	0	0	1	1
$[0,1,1]$	0	1	1	0	0	1	1	0
$[1,0,0]$	0	0	0	0	1	1	1	1
$[1,0,1]$	0	1	0	1	1	0	1	0
$[1,1,0]$	0	0	1	1	1	1	0	0
$[1,1,1]$	0	1	1	0	1	0	0	1

we can make a table for all possible codewords

x	$Had(x)$
$[0,0,0]$	$[0,0,0,0,0,0,0,0]$
$[0,0,1]$	$[0,1,0,1,0,1,0,1]$
$[0,1,0]$	$[0,0,1,1,0,0,1,1]$
$[0,1,1]$	$[0,1,1,0,0,1,1,0]$
$[1,0,0]$	$[0,0,0,0,1,1,1,1]$
$[1,0,1]$	$[0,1,0,1,1,0,1,0]$
$[1,1,0]$	$[0,0,1,1,1,1,0,0]$
$[1,1,1]$	$[0,1,1,0,1,0,0,1]$

.

For the punctured version we instead take

x	$x\cdot[1,0,0]$	$x\cdot[1,0,1]$	$x\cdot[1,1,0]$	$x\cdot[1,1,1]$
$[0,0,0]$	0	0	0	0
$[0,0,1]$	0	1	0	1
$[0,1,0]$	0	0	1	1
$[0,1,1]$	0	1	1	0
$[1,0,0]$	1	1	1	1
$[1,0,1]$	1	0	1	0
$[1,1,0]$	1	1	0	0
$[1,1,1]$	1	0	0	1

.

This construction gives us the $[4,3]$-Hadamard code.]

5. Find the generator matrix for the Hadamard and punctured Hadamard code from the list of codewords. [Answer: We form a matrix from the rows correspoding to Had($[1,0,0]$), Had($[0,1,0]$), and Had($[0,0,1]$), to get $G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$ for the Hadamard code and $G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ for the punctured version.]

6. Find the dual code for the [8,4] Hadamard code. Express your answer though the row reduced form of the generator matrix for this dual.

[Answer: It is not possible to replace G with $[I|P]$ for any matrix P as the closest we can get through row reduction is $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$. Working with this matrix gives us a nullspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Putting these together and taking the transpose gives us $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$. Finally we can row reduce this to arrive at $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.]

7. Find the dual code for the [4,3] punctured Hadamard code.

   [Answer: If we begin by writing the generator matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ in row reduced form as $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, we can then take $[-P|I]$ which is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.]

8. Use Sylvester's construction to find the [4,2]-Hadamard code.

   [Answer: We begin with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and then form the matrix $\begin{bmatrix} A & A \\ A & A' \end{bmatrix}$ where A' is A with the entries reversed. This gives us $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. We can read these off to arrive at a table

x	$Had(x)$
$[0,0]$	$[0,0,0,0]$
$[0,1]$	$[0,1,0,1]$
$[1,0]$	$[0,0,1,1]$
$[1,1]$	$[0,1,1,0]$

.

9. Use Sylvester's construction to find the [8,4]-Hadamard code.

   [Answer: We use the same process as before to taking $\begin{bmatrix} A & A \\ A & A' \end{bmatrix}$ where A is now $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

This gives us

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We can use this to directly form a table

x	$Had(x)$
$[0,0,0]$	$[0,0,0,0,0,0,0,0]$
$[0,0,1]$	$[0,1,0,1,0,1,0,1]$
$[0,1,0]$	$[0,0,1,1,0,0,1,1]$
$[0,1,1]$	$[0,1,1,0,0,1,1,0]$
$[1,0,0]$	$[0,0,0,0,1,1,1,1]$
$[1,0,1]$	$[0,1,0,1,1,0,1,0]$
$[1,1,0]$	$[0,0,1,1,1,1,0,0]$
$[1,1,1]$	$[0,1,1,0,1,0,0,1]$

.]

10. Use Sylvester's construction to find the $[8,5]$-punctured Hadamard code.

[Answer: We form $\begin{bmatrix} A \\ A' \end{bmatrix}$ where A was the final matrix in the $[8,4]$ construction to get the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

If we wish, we can use this to directly form a table

x	$Had(x)$
$[0,0,0,0]$	$[0,0,0,0,0,0,0,0]$
$[0,0,0,1]$	$[0,1,0,1,0,1,0,1]$
$[0,0,1,0]$	$[0,0,1,1,0,0,1,1]$
$[0,0,1,1]$	$[0,1,1,0,0,1,1,0]$
$[0,1,0,0]$	$[0,0,0,0,1,1,1,1]$
$[0,1,0,1]$	$[0,1,0,1,1,0,1,0]$
$[0,1,1,0]$	$[0,0,1,1,1,1,0,0]$
$[0,1,1,1]$	$[0,1,1,0,1,0,0,1]$
$[1,0,0,0]$	$[1,1,1,1,1,1,1,1]$
$[1,0,0,1]$	$[1,0,1,0,1,0,1,0]$
$[1,0,1,0]$	$[1,1,0,0,1,1,0,0]$
$[1,0,1,1]$	$[1,0,0,1,1,0,0,1]$
$[1,1,0,0]$	$[1,1,1,1,0,0,0,0]$
$[1,1,0,1]$	$[1,0,1,0,0,1,0,1]$
$[1,1,1,0]$	$[1,1,0,0,0,0,1,1]$
$[1,1,1,1]$	$[1,0,0,1,0,1,1,0]$

.]

7.3 Automorphism Groups of Codes

1. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.
[Answer: The codewords are $[0,0,0]$ and $[1,1,1]$ and these are fixed by all permutations of columns. Thus our group isomorphic to S_3.]

2. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
[Answer: The non-zero codewords are the complete set of all length three binary strings which have one zero and two ones. As a permutation cannot change the number of ones and zeros, every codeword is fixed by every permutation of columns. Thus we get a group isomorphic to S_3.]

3. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
[Answer: No codeword can have a nonzero entry in the second slot, so any permutation mapping anything but two to two cannot be included. We get $\{e, (1,3)\}$ which is isomorphic to S_2.]

4. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$.
[Answer: Our codewords are $[0,0,0,0], [1,1,0,0]$ with the former being fixed by any possible permutation. The permutations in the automorphism group must therefore send the latter to itself. We get $\{e, (12), (34), (12)(34)\}$ as the four permutations that accomplish this. This is a group of order four with no element of order four and thus our automorphism group must be isomorphic to the Klein four group.

5. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$.
[Answer: Our codewords are $[0,0,0,0]$, and $[1,1,1,0]$. The latter is fixed by anything leaving the fourth column fixed. Thus our automorphism group must be isomorphic to S_3.

6. Find the automorphism group of the binary code given by generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

   [Answer: Our codewords are $[0,0,0,0], [1,1,1,1], [1,0,0,1]$ and $[0,1,1,0]$. We want to find all permutations which map the set $\{[1,0,0,1],[0,1,1,0]\}$ to itself. Note that $\{e,(1,4),(2,3),(1,4)(2,3)\}$ forms the set of permutations which leave either word fixed. Consider the collection of permutations taking the first word to the second. This set is $\{(12)(34),(13)(24),(1243),(1342)\}$. As these also take the second word back to the first we know our automorphism group is $\{e,(1,4),(2,3),(1,4)(2,3),(12)(34),(13)(24),(1243),(1342)\}$. From the orders of our elements, and the fact that this group is not abelian, we can conclude that this group is isomorphic to D_4.]

7. Find the automorphism group of the code given by $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

   [Answer: Our code words are $[1,0,0,1],[0,1,1,1],[1,1,1,0]$ and $[0,0,0,0]$. The first word can only go itself, as no permutation will change the number of ones. Thus the only permutations that could be in $Aut(C)$ are $(14),(23),(14)(23)$ and e. These all interchange code words $[0,1,1,1]$ and $[1,1,1,0]$, and they all fix $[0,0,0,0]$ so they are all in $Aut(C)$.]

8. Find the automorphism group of the [4,3]-punctured Hadamard code.

   [Answer: After row reduction, we see that this code is given by generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.
   This generates the elements $[1,1,1,1]$, $[0,0,0,0]$ and the six elements with exactly two ones and two zeros. As every permutation fixes the numbers of ones and zeroes, the automorphism group must be all of S_4.]

9. Is $(13)(57)$ in the automorphism group for the Hamming code given by $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$?

   [Answer: Yes. Every codeword must be a linear combination of rows so we only need show that it maps rows to codewords. Row one becomes row three, row three becomes row one, and rows two and four are fixed by this element of S_7. This shows it is in $Aut(C)$.]

10. Is (1234567) in the automorphism group for the Hamming code given by $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$?

    [Answer: No. This group element in S_7 maps the codeword $[1,0,0,0,1,1,0]$ to $[0,1,0,0,0,1,1]$ which is not a linear combination of any of the rows. If we try writing it as $ar_1 + br_2 + cr_3 + dr_4$ for some a,b,c,d, we need $b=1$ but $a=c=d=0$ due to the entries in the first four columns. This gives us r_2 which does not equal our word.]

11. Is $(13)(57)$ in the automorphism group for the Hamming code given by $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$?

    [Answer: Yes. Every codeword must be a linear combination of rows so we only need show that it maps rows to codewords. Row one becomes row two, row two becomes row three, row three becomes row one, and row four is fixed by this element of S_7. This shows it is in $Aut(C)$.]

12. Is (17)(26)(35) in the automorphism group for the Hamming code given by $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$?

    [Answer: No. This group element in S_7 maps the codeword $[1,0,0,0,1,1,0]$ to $[0,1,1,0,0,0,1]$ which is not a linear combination of any of the rows. If we try writing it as $ar_1 + br_2 + cr_3 + dr_4$ for some a, b, c, d, the first four columns imply we need $b = c = 1$ and $a = d = 0$. This gives us $[0,1,1,0,1,1,0]$ which does not equal our word.]

13. Is (23)(56) in the automorphism group for the Hamming code given by $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$?

    [Answer: Yes. Every codeword must be a linear combination of rows so we only need show that it maps rows to codewords. Row one and row four are fixed by this element of S_7 and rows two and three are interchanged. This shows our element is in $Aut(C)$.]

14. Is (1234685) in the automorphism group for the extended Hamming code given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}?$$

    [2]

    [Answer: Yes. Every codeword is a combination of the four rows of this matrix. If the permutation sends these rows to codewords, then it sends code words to codewords. We examine what happens to the four rows under (1234685)

    Row 1 becomes $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$. This is simply row two of the matrix, hence a codeword.

    Row 2 becomes $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. This is the sum of rows one and three.

    Row 3 becomes $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$. This is the sum of rows one and four.

    Row 4 becomes $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. This is simply row one. As (1234685) maps code words to code words, this shows it is in the automorphism group.]

15. Is (15)(26)(37)(48) in the automorphism group for the extended Hamming code given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}?$$

[2] Thanks for this and the next two examples: "Build a Sporadic Group in Your Basement" by P. Becker, M. Derka, S Houghten and J. Ulrich, American Mathematical Monthly, v.124 no. 4 April 2017

[Answer: Yes. Again, we show it sends rows to codewords.

Row 1 becomes $\begin{bmatrix}0 & 1 & 1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$. This is the sum of rows two, three and four.

Row 2 becomes $\begin{bmatrix}1 & 0 & 1 & 1 & 0 & 1 & 0 & 0\end{bmatrix}$. This is the sum of rows one, three and four.

Row 3 becomes $\begin{bmatrix}1 & 1 & 0 & 1 & 0 & 0 & 1 & 0\end{bmatrix}$. This is the sum of rows one, two and four.

Row 4 becomes $\begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$. This is the sum of rows one, two, and three. As (15)(26)(37)(48) maps code words to code words, this shows it is in the automorphism group.]

16. Is (1234)(5678) in the automorphism group for the extended Hamming code given by

$$G = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\end{bmatrix}?$$

[Answer: Yes. Again, we show it sends rows to codewords. Our element sends row one to row two, row two to row three, row three to row four, and row four back to row one. As (1234)(5678) maps all codewords to codewords, this shows it is in the automorphism group.]

17. Is (12) in the automorphism group for the extended Hamming code given by $G = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\end{bmatrix}$?

[Answer: No. Here we show it does not always send rows to codewords.

Row 1 becomes $\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 1 & 1 & 1\end{bmatrix}$. If we try to write this as a linear combination of rows, we must have $ar_1 + br_2 + cr_3 + dr_4$ for some $a, b, c, d \in \mathbb{Z}_2$. Because 1 is in the second column, we must have $b = 1$. Because 0 is in columns one, three and four, we need $a = c = d = 0$. But then we get only row two which is not equal to the vector we are trying to represent.]

18. Find the automorphism group of the ternary code given by generator matrix $G = \begin{bmatrix}1 & 2\end{bmatrix}$.
[Answer: Our codewords are $[0, 0], [1, 2]$ and $[2, 1]$. Here there are only two permutations to check. (12) fixes $[0, 0]$ and switches the others, and e fixes everything. Thus both are in $Aut(C)$ and it is isomorphic to $\mathbb{Z}_2$.]

19. Find the automorphism group of the ternary code given by generator matrix $G = \begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}$.
[Answer: Our codewords are $[0, 0, 0, 0], [1, 2, 1, 0]$ and $[2, 1, 2, 0]$. Each has a different number of zeros (or ones or twos) so no permutation can map one to a combination of others. Thus any element of $Aut(C)$ must map each to itself. The only permutations doing this are (13) and e, so $Aut(C) = \{e, (13)\}$ and is isomorphic to $\mathbb{Z}_2$.]

20. Find the automorphism group of the ternary code given by generator matrix $G = \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\end{bmatrix}$.

[Answer: We now have nine codewords, so we have to be more careful. We will use the fact that elements in $Aut(C)$ must send rows of the generator matrix to codewords, which are combinations of rows. Notice that (123) maps $[1, 0, 2]$ to $[2, 1, 0]$ which is twice row one plus row two. It sends row two to twice row one. Thus $(123) \in Aut(C)$. This implies $(123)^2 = (132)$ must be in $Aut(C)$. Thus we know $Aut(C)$ contains a subgroup of order three, and thus by Lagrange, is either $\{e, (123), (132)\}$ or all S_3. Note that (12) maps $[1, 0, 2]$ to $[0, 1, 2]$ which is row two, and also maps row two to row one. This is all we needed to check. Since $|Aut(C)| > 3$ we know it is all of S_3.]

21. Find the automorphism group of the ternary code given by generator matrix $G = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

    [Answer: Again we have nine codewords, so we examine where elements of S_3 send our rows. Notice that (123) sends $[1, 0, 2]$ to $[2, 1, 0]$. If $[2, 1, 0] = ar_1 + br_2$ we know a must be two and b must be one. That gives $[2, 1, 2]$. This shows (123) is not a codeword. It also shows (132) is not a codeword, as that would imply (123) was due to closure.

    We know our group contains no 3-cycles, so it cannot contain two 2-cycles. It must either be $\{e\}$ or isomorphic to $\mathbb{Z}_2$. Notice that (13) sends $r_2 = [0, 1, 1]$ to $[1, 1, 0]$. If we try to write this as $ar_1 + br_2$ we get $a = 1$ and $b = 1$, does indeed give us our codeword. It also sends $r_1 = [1, 0, 2]$ to $[2, 0, 1]$ which is twice r_1. Thus it will send all codewords to codewords and is in $Aut(C)$. Since there can be at most one 2-cycle, there can be nothing else in $Aut(C)$. This shows us that $Aut(C) = \{e, (13)\}$ and thus is isomorphic to $\mathbb{Z}_2$.]

Chapter 8

Acknowledgments

The author would like to thank Summer Pirro, Brandon Jacome, and Matt Brauer from the Applied Algebra class of Spring 2017.